action at each layer. These parameters are chosen such that a favorable tradeoff between operating costs and performance level achieved is obtained. An approximate tradeoff measure is developed to ease the computational burden required to carry out the tradeoff analysis.

The investigations reported in the preceding sections represent an initial step in the formal treatment of this multilayer strategy and its associated tradeoff problem. There is, therefore, a considerable need for further research in this area.

References

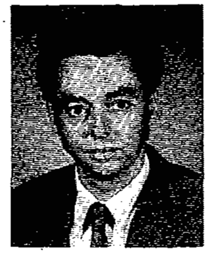
[1] I. Lefkowitz, "Multilevel approach applied to control system design," *Trans. ASME*, vol. 88, June 1966.

[2] W. Findeisen and I. Lefkowitz, "Design and applications of multilayer control," in *Proc. 4th IFAC Congr.*, Warsaw, Poland, 1969.

[3] J. F. Donoghue, "Economic tradeoffs for multilayer control systems," Ph.D. dissertation, Syst. Res. Cent., Case Western Reserve Univ., Cleveland, Ohio, Oct. 1970.

[4] R. A. Rohrer and M. Sobral, Jr., "Sensitivity considerations in optimal system design," *IEEE Trans. Automat. Contr.*, vol. AC-10, pp. 43–48, Jan. 1965.

[5] G. Hadley and T. M. Whitin, *Analysis of Inventory Systems.* Englewood Cliffs, N. J.: Prentice-Hall, 1963.



John F. Donoghue (S'60–M'70) was born in New York, N. Y., on February 6, 1940. He received the B.S. and M.S. degrees in electrical engineering from Northeastern University, Boston, Mass., in 1963 and 1965, respectively, and the Ph.D. degree in control engineering from Case Western Reserve University, Cleveland, Ohio, in 1970.

He is currently with the Industrial Nucleonics Corporation, Columbus, Ohio, and is engaged in the design of computer control systems for the basic industries.

Dr. Donoghue is a member of the Operations Research Society of America, Sigma Xi, Tau Beta Pi, and Eta Kappa Nu.

Irving Lefkowitz (M'67) received the B.S. degree in chemical engineering from Cooper Union School of Engineering, New York, N. Y., and the M.S. and Ph.D. degrees in control engineering from Case Institute of Technology, Cleveland, Ohio.

He is presently Professor of Engineering at Case Western Reserve University, Cleveland, Ohio, and Director of the research program in the control of complex systems.

Team Decision Theory and Information Structures in Optimal Control Problems—Part I

YU-CHI HO, SENIOR MEMBER, IEEE, AND K'AI-CHING CHU, MEMBER, IEEE

***Abstract*—Information structures of organizations are studied and applied to problems of dynamic team decisions. For a causal system it is shown that there is a partially ordered precedence relation existing among the decision makers.**

The team decision problem with linear information structure and quadratic payoff function is dealt with. The primitive random variables are assumed to be jointly Gaussian. The optimal solutions for the teams in which precedents' informtion is available for the followers are obtained. It is shown that the well-known linear-quadratic-Gaussian stochastic control problem and static team decision problem are special cases of the structure considered.

I. Introduction and Problem Statement

Within a general organization there are many members, each controlling different action or decision variables at different times, each having access to different information, and each attempting to attain different goals. A *team* is an organization in which there is a single goal or payoff common to all the members. Let us consider a team composed of $i \in I = \{1, 2, \cdots, N\}$ members. Each member receives certain information z_i and controls the decision variable u_i, where the nature of these variables will be defined presently. We denote the payoff function for all the members as

$$J = J(\gamma_1, \gamma_2, \cdots, \gamma_N), \tag{1}$$

where γ_i is the control law or decision rule,

$$u_i = \gamma_i(z_i) \text{ and } \gamma_i \in \Gamma_i \tag{2}$$

used by the ith member and Γ_i is the class of admissible control laws for i. The team-theoretic optimization problem can then be informally stated as follows.

Problem 1: Find $\gamma_i^* \in \Gamma_i$ for all i such that $J(\gamma_1, \gamma_2, \cdots, \gamma_N)$ is minimized. If the system or organization evolves dynamically in time and the decisions of the members interact with the payoff as well as the information received by the members, then we have an optimal control problem in which team decision theory and information structure plays a decisive role. The purpose of this paper is to investi-

Manuscript received April 19, 1971. Paper recommended by P. P. Varaiya, past chairman of the IEEE S-CS Large Systems, Differential Games Comittee. This work was supported in part by the Joint Services Electronics Program under Contract N00014-67-A-0298-0006 through Harvard University and in part by the Guggenheim Foundation through a grant to Y.-C. Ho.

Y.-C. Ho is with the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass. 02138.

K.-C. Chu was with the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass. He is now with Systems Control, Inc., Palo Alto, Calif. 94306.

gate one such class of problems and to present various explicit results.

Marschak [1] first formulated the team problem more than 16 years ago. So far, theoretic work is mainly limited to the static teams [2], [3] in which information z_i is only the function of some random variable ξ but is independent of what other members have done. However, in a dynamic team, present information is affected by what has been done in the past. Therefore, the present estimation and decision are dependent on the actions of the other members in the past; this very dependence is itself affected by the past actions which are part of the solution to be obtained. Because of the difficulties caused by the interactions among information estimation and control variables, there has been no substantial work done in the dynamic case.

A. *Information Structures*

Let $\xi \in R^n$ denote a random vector defined on an underlying probability space $(R^n, \mathfrak{F}, P)$, and let it represent all the uncertainties of the external world which are not controlled by any of the members. We assume that the probability distribution of ξ is known to all the members and is Gaussian $N(\mathbf{0}, X)$ with $X > 0$.

The information z_i each member receives includes everything available as knowledge to him for making decisions. This consists of what he has remembered, what he has observed, and what other members have communicated to him, etc. The information z_i is assumed to be a known linear function in R^{q_i} of ξ and some of the control actions other members have taken, i.e.,

$$z_i = H_i\xi + \sum_j D_{ij}u_j, \qquad \forall\ i, \tag{3}$$

where H_i and D_{ij} are matrices of appropriate dimensions and are known to all the members. We shall be interested in only real causal systems where what happens in the future cannot affect what is observed now. Thus, in (3) we assume

$$D_{ij} \neq 0 \Longrightarrow D_{ji} = 0, \qquad \forall\ i, j \in I; \tag{4}$$

i.e., if the control action of j affects the information i, then u_i cannot affect the information of j. We have in mind here essentially a discrete-time dynamic situation in which current actions can affect, at most, information in the succeeding, but not the current, stage.

We formalize this in the following definitions.

Definition 1: We say j is *related* to i, jRi, if $D_{ij} \neq 0$.

Definition 2: We say j is a *precedent* of i, $j \prec i$, iff (a) jRi or (b) there exists distinct $r, s, t, \cdots, k \in I$ such that jRr and rRs and $sRt, \cdots, kRi$.

Graphically, we can represent the idea of precedence in a *precedence diagram.* Each member of the team [decision maker (DM)] is represented by a node so placed that the node for j is above that for i if $j \prec i$. One then draws a directed segment from node j to node i if jRi *and* there exists no k such that $j \prec k$ and $k \prec i$. A *path*, consisting of a connected series of directed segments, exists between j and i if and only if $j \prec i$.

Fig. 1.

Example 1:

$$D_{ij} = 0, \qquad \forall\ i \text{ and } i$$
$$z_i = H_i\xi, \qquad \forall\ i. \tag{5}$$

The precedence diagram for this system is simply isolated nodes. (See Fig. 1.) Since there is no explicit causal relation between the control and information of different members, we call structures such as this, with isolated members in the precedence diagram, *static teams.*

In a static team the information of each member may not be obtained at the same time, nor need the control actions executed by each member take place simultaneously. As long as there are no casual precedence relations among the members, the actual time instants when the observation and actions occur are not important.

Example 2: In a classical multistage stochastic control problem, the dynamic equation is

$$x_{i+1} = Fx_i + Gu_i + w_i, \qquad i = 1, \cdots, N, \tag{6}$$

and the observation at each stage is

$$y_i = Hx_i + v_i, \qquad i = 1, \cdots, N, \tag{7}$$

where x_i is the state, u_i the control, and w_i and v_i the independent sequences of random variables. The distributions of x_1, w_i, and v_i for all i are known to be independent zero-mean and Gaussian. The system is assumed to be of *perfect memory* in the sense that at time i, the decision maker remembers perfectly what he has known and what he has done before. Imagine there are N members of a team to control the system and that the ith member is responsible for u_i at the ith stage.

Since

$$\begin{aligned} x_i &= \text{linear in } (x_{i-1}, u_{i-1}, w_{i-1}) \\ &= \text{linear in } (x_1; u_1, \cdots, u_{i-1}; w_1, \cdots, w_{i-1}), \qquad \forall\ i, \end{aligned} \tag{8}$$

which is clear from the recursive equation (6), the information for member i is

$$z_i = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{i-1} \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_i \end{bmatrix} = \text{linear in } (x_1; u_1, \cdots, u_{i-1}; w_1, \cdots, w_{i-1}; v_1, \ldots, v_i), \qquad \forall\ i. \tag{9}$$

The random variables $x_1; w_1, \cdots, w_N; v_1, \cdots, v_N$ all together represent the same thing as ξ defined earlier, that

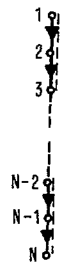


Fig. 2.

$$\xi^T = (x_1^T;\ w_1^T,\ \cdots,\ w_N^T;\ v_1^T,\ \cdots,\ v_N^T). \quad (10)$$

Equation (10) can be rewritten as

$$z_i = \text{linear in } (\xi, u_1, \cdots, u_{i-1})$$
$$= H_i\xi + \sum_{j=1}^{i-1} D_{ij}u_j \quad (11)$$

for some H_i and D_{ij} and for all i, where none of the matrices D_{ij} are zero matrices. We note from (9) that z_j is imbedded in z_i as components if $j < i$. We stress this fact by drawing a *memory–communication line* segment (dotted line) from j to i on the precedence diagram. Intuitively, this suggests that whatever j knows is either remembered by i (in the case of one player acting as a different DM at different times) or is passed on to i (when we have different players). The precedence diagram with its memory–communication line for this example is shown in Fig. 2. Note, since z_j includes z_{j-1}, it is not necessary to have a dotted-line segment joining nodes $j+1$ and $j-1$.

The precedence diagram with its memory–communication lines will be called the *information structure diagram.* It is a graphic representation of (3). The information structure diagram is essential to the analysis of information transmission and causal relations. Any linear dynamic system of (6) and (7) (time varying or not) can be put in our normalized form of (3) by a method similar to that of Example 2. Linear dynamic processes without perfect memory or with only partial feedback fit naturally into our structure. A general example of a linear-Gaussian team problem is found in Example 3.

Example 3:

$$z_1 = H_1\xi$$
$$z_2 = H_2\xi$$
$$z_3 = H_3\xi + D_{31}u_1 + D_{32}u_2$$
$$z_4 = \begin{bmatrix} H_4' \\ \\ H_3 \end{bmatrix}\xi + \begin{bmatrix} 0 & 0 & D_{43}' \\ & & \\ D_{31} & D_{32} & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$
$$z_5 = D_{54}u_4$$
$$z_6 = H_6\xi + D_{64}u_4$$
$$z_7 = H_7\xi$$
$$z_8 = D_{87}u_7. \quad (12)$$

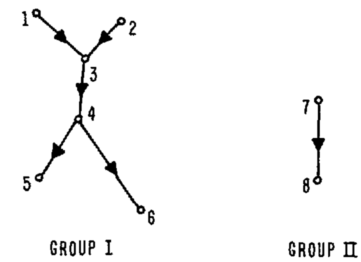


Fig. 3.

The information structure diagram is displayed in Fig. 3. Members one, two, and seven are *starting decision makers* of the team; members five, six, and eight are the *terminating decision makers.* In the sequel we shall index the members in such a way that if j is a precedent of i, then $j < i$.

B. Control Laws

Each decision maker makes a decision at a single time moment. The information z_i is made available for the ith member just before he makes his decision. In practice someone may have to make a decision more than once at different times, then either the information available on all these occasions is the same, and then these decisions are considered as a single one picked from a product set, or else the information available is not the same, and then one can assume separate members for each occasion.

We define the class of admissible control laws for the ith DM, Γ_i, as the set of all Borel-measurable functions γ_i: $R^{q_i} \to R^{k_i}$. Note that for fixed $\gamma_i \in \Gamma_i$, $i = 1, \cdots, N$, (3) induces for each i a sub-σ-algebra $\mathcal{Z}_i \subset \mathfrak{F}$, and z_i are well-defined random variables measurable with respect to $\mathcal{Z}_i$. Let u_i take value in $U_i = R^{k_i}$, then we have a σ-algebra $\mathfrak{F}_i$ on U_i such that $\gamma_i^{-1}(\mathfrak{F}_i) = \mathcal{Z}_i$. Note that with the exception of the static team, Example 1, $\mathfrak{F}_i$, $\mathcal{Z}_i$ $\forall\ i$, are dependent on the choice of $\gamma = [\gamma_1, \cdots, \gamma_N]$. Therein lie the major difficulties of the solution of dynamic team problems. Fortunately, for a large class of such problems with special information structures, this difficulty can be circumvented.

C. Payoff Function

The common goal for all members is to minimize the function

$$J(\gamma_1, \cdots, \gamma_N) = E[\mathcal{J}] = E[\tfrac{1}{2}\,\boldsymbol{u}^TQ\boldsymbol{u} + \boldsymbol{u}^TS\xi + \boldsymbol{u}^Tc],$$
$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_N \end{bmatrix}, \quad (13)$$

where Q is symmetric positive definite and u_i are given by (2) and the expectation is taken with respect to the *a priori* ξ. Matrices Q, S and vector c are of appropriate dimensions and are known to all the members. As stated earlier, with the particular choice for the class of admissible control laws, all u_i are well-defined random variables and the meaning of the expectation is clear. Problem 1 is well defined.

II. Necessary Conditions for Optimality

Problem 1: This problem is stated in the so-called "normal form."[1] From a practical viewpoint, Problem 1 is not in a suitable form for explicit solution except in the case where the cardinality of each admissible control set is finite and very small. To put it in a more useful form, we first consider a relaxed version of the problem.

Problem 2: Find $\gamma_i^* \in \Gamma_i$ for all i such that

$$J(\gamma_1^*, \cdots, \gamma_{i-1}^*, \gamma_i^*, \gamma_{i+1}^*, \cdots, \gamma_N^*) \leq J(\gamma_1^*, \cdots, \gamma_{i-1}^*, \gamma_i, \gamma_{i+1}, \cdots, \gamma_N^*) \quad (14)$$

for all $\gamma_i \in \Gamma_i$ and for all i.

Optimality relation (14) is certainly a necessary, but not sufficient, condition for the global solution of Problem 1. In other words, Problem 2 is a noncooperative version of Problem 1 in the language of game theory. We call the solution of Problem 2 *member-by-member optimal*. Next we rewrite the expectation in (13) as

$$\begin{aligned} J &= E\{\mathcal{J}[\gamma_1^*(z_1), \cdots, \gamma_{i-1}^*(z_{i-1}), \gamma_i(z_i), \gamma_{i+1}^*(z_{i+1}), \cdots, \gamma_N^*(z_N), \xi]\} \\ &= E_{z_i}\{E[\mathcal{J}|z_i]\} \text{ for fixed } \gamma_1^*, \cdots, \gamma_{i-1}^*, \gamma_i, \gamma_{i+1}^*, \cdots, \gamma_N^*, \end{aligned} \quad (15)$$

where the second expectation is taken with respect to any *given values* of z_i. However, for *fixed* control laws $\gamma_1^*, \cdots, \gamma_{i-1}^*, \gamma_{i+1}^*, \cdots, \gamma_N^*$ and *any fixed* z_i,

$$\min_{\gamma_i \in \Gamma_i} E_{z_i}\{E[\mathcal{J}|z_i]\} \leftrightarrow \min_{u_i} E[\mathcal{J}(\gamma_1^*(z_1), \cdots, u_1 = \gamma_i(z_i), \cdots, \gamma_N^*(z_N), \xi)|z_i], \qquad \forall z_i. \quad (16)$$

Problem 3: We come to the third version of the team optimization problem, i.e.,

$$\min_{u_i} E[\tfrac{1}{2}\, \boldsymbol{u}^T Q \boldsymbol{u} + \boldsymbol{u}^T S \xi + \boldsymbol{u}^T c|z_i] \triangleq \min_{u_i} J_i$$

where

$$\boldsymbol{u}^T = [\gamma_1^{*T}(z_1), \cdots, \gamma_{i-1}^{*T}(z_{i-1}), \gamma_i^T(z_i), \gamma_{i+1}^{*T}(z_{i+1}), \cdots, \gamma_N^{*T}(z_N)]$$

for fixed $\gamma_j^*(z_j)$, $j \neq i$, and any z_i.

Problem 3 is in the so-called "extensive form" or "seminormalized form" in the sense that all except one control variable are given as strategies. A necessary condition for the optimality of Problem 3 can be obtained by taking partial derivatives of J_i with respect to u_i. However, the partial differentiation must be considered with care. Terms involve u_j such that $j \succ i$ must be included since $u_j = \gamma_j^*(z_j)$ and z_j depends on u_i through the causal relation. Thus in general, the partial differentiation will result in

$$\begin{aligned} Q_{ii}\gamma_i &+ \sum_{j\neq 1} Q_{ij}E(\gamma_j|z_i) + S_iE(\xi|z_i) + c_i \\ &+ \sum_{j \succ i}\Big[u_i^T Q_{ij}\frac{\partial}{\partial u_i}E(\gamma_j|z_i) + \frac{\partial}{\partial u_i}E(\gamma_j^T S_j \xi|z_i) \\ &+ \frac{\partial}{\partial u_i}E(\gamma_j^T|z_i)c_j + \sum_{k\neq i}\frac{\partial}{\partial u_i}E(\gamma_k^T Q_{kj}\gamma_j|z_i)\Big] = 0 \end{aligned} \quad (17)$$

for all values of z_i and for all $i = 1, \cdots, N$ where Q_{ij} means the ith row- and jth column-partitioned block of Q, S_j, the jth row-partitioned block of S, c_i, and the ith partitioned subvector of c. The last four partial derivative terms of (17) depend explicitly on the form of controls of the following members. This is rather unsatisfactory inasmuch as we are attempting to solve for those control laws through the consideration of (17). Now we see more explicitly the difficulties connected with the solution of dynamic team optimization problems. Furthermore, because of this causal dependence of γ_j on γ_i for $j \succ i$, the payoff function in Problem 3 is not generally convex in γ_i even though it is quadratic in $\boldsymbol{u}$ with $Q > 0$, nor are z_i generally Gaussian even though ξ are given as Gaussian. The quadratic-Gaussian nature of J and z is *dependent* on the form of the control laws γ_i chosen. Fortunately, for a variety of classes of problems these difficulties can be overcome or contained in one way or another. In later sections and in Part II of this paper these solutions will be discussed.

III. Static Teams

The results of this section are essentially those first obtained by Radner [2]. We shall derive them in terms of our notation, as they will be needed in later sections.

For this problem each member's information is a linear function of ξ only, i.e., $D_{ij} = 0$ in (3) and

$$z_i = H_i\xi, \qquad \forall i. \quad (18)$$

to avoid triviality, we assume H_i is $q_i \times n$ with $n > q_i$ and that it is of maximal rank.

The necessary condition of (17) immediately simplifies to

$$Q_{ii}\gamma_i(z_i) + \sum_{j\neq i} Q_{ij}E(\gamma_j(z_j)|z_i) + S_iE(\xi|z_i) + c_i = 0, \qquad \forall z_i \text{ and } \forall i, \quad (19)$$

since all partial derivative terms in (17) are zero. Furthermore, we have the following lemma.

Lemma: The optimal solution of J of (13) is unique in static teams.

Proof: Consider $\gamma_i = \gamma_i^o + \epsilon\,\delta_i$. Since z_j is independent of γ_i for all j and i, J is strictly convex and quadratic in all γ_i. This, together with the dominated convergence theorem, enables one to interchange expectation and differentiation [2] and to show $(\partial^2 J/\partial\epsilon^2) = E(\delta_i^T Q_{ii}\delta_i) > 0$. This shows that J is strictly convex in ϵ and the conclusion follows.

Consequently, any solution of the necessary conditions

[1] It is stated in accordance with the usually accepted meaning in game theory [7] or Bayesian statistical decision theory [8].

(19) by uniqueness will be the global optimal solution. We try the control laws of the form

$$u_i = \gamma_i(z_i) = A_iz_i + b_i, \qquad \forall\ i, \tag{20}$$

where A_i and b_i are $k_i \times q_i$-matrix and k_i-vector, respectively, to be determined. Substituting (20) into (19),

$$Q_{ii}(A_iz_i + b_i) + \sum_{j\neq i} Q_{ij}E(A_jH_j\xi + b_j|z_i) + S_iE(\xi|z_i) + c_i = 0$$

$$Q_{ii}(A_iz_i + b_i) + \Big[\sum_{j\neq i} Q_{ij}A_jH_j + S_i\Big] \cdot E(\xi|z_i) + \sum_{j\neq i} Q_{ij}b_j + c_i = 0$$

$$\Big(\sum_{j=1}^{N} Q_{ij}b_j + c_i\Big) + \Big[Q_{ii}A_i + \Big(\sum_{j\neq i} Q_{ij}A_jH_j + S_i\Big) \cdot XH_i^T(H_iXH_i^T)^{-1}\Big]z_i = 0 \tag{21}$$

for all z_i and for all i, where we have utilized simple properties of jointly Gaussian random variables to evaluate $E(\xi|z_i)$ and the fact that $H_iXH_i^T$ has an inverse. Since (21) must be true for all z_i, we get

$$(b_1^T, b_2^T, \cdots, b_N^T) = c^TQ^{-1} \tag{22}$$

and

$$Q_{ii}A_i + \sum_{j\neq i} Q_{ij}A_jH_jXH_i^T(H_iXH_i^T)^{-1} = -S_iXH_i^T(H_i^TXH_i^T)^{-1}, \qquad \forall\ i,$$

or

$$\sum_j Q_{ij}A_j(H_jXH_i^T) = -S_iXH_i^T, \qquad \forall\ i. \tag{23}$$

The coefficients of the elements of A_i in the linear simultaneous equation (23) form a positive-definite matrix [2, p. 870, lemma]; hence the elements of all A_i are uniquely solvable from (23). Thus, we have the following theorem.

Theorem (Radner): The control law of (20) with (22) and (23) is optimal for the static-team optimization problem with the information structure (18).

IV. Dynamic Team with Partially Nested Information Structure

In this section we wish to study a special class of information structure motivated by the following informal consideration. Suppose, for each DM i and all his precedent j, the information variables z_j can be generated from z_i in the sense that knowing z_i implies knowing z_j. A particular case of such information structure is that of Example 2 of Section I. In other words, the memory–communication structure is the *same* as the precedence relation in the information structure diagram. Such structure has the property that the action of all the precedents is completely determined once the control laws are fixed. Thus the only random effects in z_i are due to the structure of the external world ξ, which is *not* solution or control law dependent. We define such structure formally.

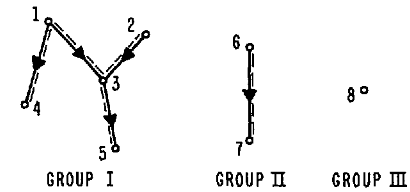


Fig. 4.

Definition 3: An information structure (3) is called *partially nested* if $j \prec i$ implies $Z_j \subset Z_i$ for all i, j, and $\gamma \in \Gamma$.

Example 1:

$$z_1 = H_1\xi$$

$$z_2 = H_2\xi$$

$$z_3 = \begin{bmatrix} H_1 \\ H_2 \\ H_3' \end{bmatrix}\xi + \begin{bmatrix} 0 \\ 0 \\ D_{31}' \end{bmatrix}u_1 + \begin{bmatrix} 0 \\ 0 \\ D_{32}' \end{bmatrix}u_2$$

$$z_4 = \begin{bmatrix} H_1 \\ H_4' \end{bmatrix}\xi + \begin{bmatrix} 0 \\ D_{41}' \end{bmatrix}u_1$$

$$z_5 = \begin{bmatrix} H_1 \\ H_2 \\ H_3' \\ H_5' \end{bmatrix}\xi + \begin{bmatrix} 0 \\ 0 \\ D_{31}' \\ D_{51}' \end{bmatrix}u_1 + \begin{bmatrix} 0 \\ 0 \\ D_{32}' \\ 0 \end{bmatrix}u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ D_{53}' \end{bmatrix}u_3$$

$$z_6 = H_6\xi$$

$$z_7 = \begin{bmatrix} H_6 \\ H_7' \end{bmatrix}\xi + \begin{bmatrix} 0 \\ D_{76}' \end{bmatrix}u_6$$

$$z_8 = H_8\xi.$$

The information structure diagram of this example is displayed in Fig. 4. It is clear from the diagram and above information structure that what the precedents have known will always be known by their followers. For instance, member three's precedents are one and two; however, z_1 and z_2 are the first and second components of z_3, respectively.

In a system with partially nested information pattern, the follower can always deduce the action of its precedents. For example, for a fixed γ_1, the first member of Example 2 has as his control

$$u_1 = \gamma_1(z_1). \tag{24}$$

Since the team can agree in advance on the decision rule or control law used, the third member can deduce the action u_1 exactly by using (24). Thus, the extra information u_1 is redundant when the value of z_1 is already part of the third member's information. The term $D_{31}u_1$ is redundant and hence can be deleted from z_3. Likewise, term $D_{32}u_2$ is also redundant and hence can be deleted from z_3, since z_2 is the second component of z_3.

Thus, we can formulate z_3 in an equivalent way such that

$$\hat{z}_3 = \begin{bmatrix} H_1 \\ H_2 \\ H_3' \end{bmatrix} \xi. \tag{25}$$

Similarly, all the $\Sigma D_{ij}u_j$ terms attached to z_4, z_5, and z_7 can be deleted without any change in the team performance. In general, it is clear that we have the following theorem.

Theorem 1: In a dynamic team with the partially nested information structure,

$$z_i = H_i\xi + \sum_{j \{ i} D_{ij}u_j \tag{26}$$

is equivalent to an information structure in static form for any fixed set of control laws

$$\hat{z}_i = [\{H_j\xi | j \{ i \text{ or } j = i\}]. \tag{27}$$

Proof: We partition the N-members into the following disjoint sets:

$N_1 = \{$set of starting members$\}$
$N_2 = \{$set of members having i as precedent, where $i \in N_1\}$

$\cdot$
$\cdot$
$\cdot$

$N_j = \{$set of members having i as precedent, where $i \in N_{j-1}\}$.

It is clear that $\hat{z}_i = z_i = H_i\xi$ for all $i \in N_1$. Now let

$$\begin{aligned} \hat{z}_i &= z_i - \sum_{j \{ i} D_{ij}\gamma_j(z_j) \\ &= H_i\xi, \qquad \forall\ i \in N_2 \text{ and } j \in N_1. \end{aligned} \tag{28}$$

Since $Z_i \supset Z_j$, then $\hat{z}_i$ will be Z_i measurable. Conversely, knowing $\hat{z}_i$ we can compute

$$z_i = \hat{z}_i + \sum_{j \{ i} D_{ij}\gamma_j(\hat{z}_i), \qquad \forall\ i \in N_2 \text{ and } j \in N_1. \tag{29}$$

Now, by recursion we can calculate z_i from $\hat{z}_i$ or vice versa for $i \in N_3, N_4, \cdots,$ etc. Q.E.D.

Remark 1: The reduction of (26) to (27) is possible only when we are considering pure strategy solution exclusively. In more general game-theoretic optimization problems with different information for each member in a team and different payoff functions for each of the teams, partially nested information structure alone may not be sufficient to effect the reduction—since with mixed strategies knowing all that others have known is not sufficient to deduce what they have actually done.

Remark 2: Note that the validity of Theorem 1 is independent of the nature of the criterion function J. Furthermore, so long as some invertibility conditions are satisfied, z_i need not be linear functions of ξ and u_j for $j \{ i$; nor does ξ have to be Gaussian. The property of partial nesting only depends on the definition of the various sub-σ algebra in Definition 3.

Theorem 2: In a dynamic team with partially nested information structure, the optimal control for each member exists, is unique and linear in z_i.

Proof: As shown by Theorem 1, a team with information structure (26) is equivalent to one with static structure (27). Therefore, by Radner's theorem, the optimal control law exists, is unique and linear in $\hat{z}_i$. Since $\hat{z}_i = z_i$ for $i \in N_1$, γ_i is linear in z_i for $i \in N_1$ also. By (28) we deduce that γ_i is linear also in z_i for $i \in N_2$. Repeated application of Theorem 1 and (28) yields the desired result. Q.E.D.

Application 1—Linear Quadratic-Gaussian Control Problems: Consider again Example 2 of Section II with a quadratic payoff function

$$J = E\left[\frac{1}{2}x_{N+1}^T S_{N+1} x_{N+1} + \frac{1}{2}\sum_{k=1}^{N} (x_k^T H^T H x_k + u_k^T b u_k)\right] \tag{30}$$

where $S_{N+1} \geq 0$, $B > 0$.

After absorbing recursive dynamic relations (6) into information functions (9) and payoff function (30), we have

$$z_i = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{i-1} \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_i \end{bmatrix} = \text{linear in } (\xi,\ u_1,\ \cdots,\ u_{i-1}), \qquad \forall\ i \tag{31}$$

and

$$J = E\left[\frac{1}{2}\boldsymbol{u}^T Q \boldsymbol{u} + \boldsymbol{u}^T S\xi\right] + \text{terms independent of } \boldsymbol{u} \tag{32}$$

where $\xi^T = (x_1^T;\ w_1^T,\ \cdots,\ w_N^T;\ v_1^T,\ \cdots,\ v_N^T)$, which is Gaussian. Also $Q > 0$, Q and S are matrices dependent on only the original parameters of the problem.

Since the information z_i for different members is nested in their natural sequence, by Theorem 2 the optimal controls exist, are unique and linear in z_i such that

$$u_i^* = A_i z_i + b_i, \qquad \forall\ i \tag{33}$$

for some A_i and b_i.

In control literature the solution (33) permits further simplification in the sense that the measurement history z_i admits a pair of finite-dimensional sufficient statistics $\hat{x}_i$ (linear in y_i) and P_i which are the mean and covariance, respectively, of a Kalman–Bucy filter. u_i^* can be expressed as a linear function of $\hat{x}_i$ only. Computationally and physically, this is both meaningful and simplifying [4]. However, our purpose here is to demonstrate the intrinsic nature of the classic linear-quadratic-Gaussian (LQG) control problem. We have observed that its information structure is basically equivalent to a static one and it permits a linear solution.

Remark 3: Note that this conclusion concerning the optimality of the linear solution is *independent* of the correlations between x_1, w_i, v_j for all i, j. In fact, the various noise sequences need not even be Markov. We only require that they be jointly Gaussian distributed.

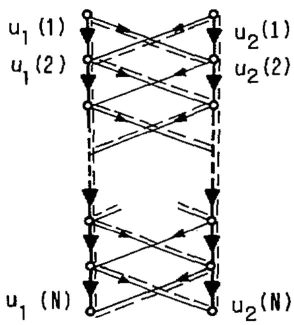


Fig. 5.

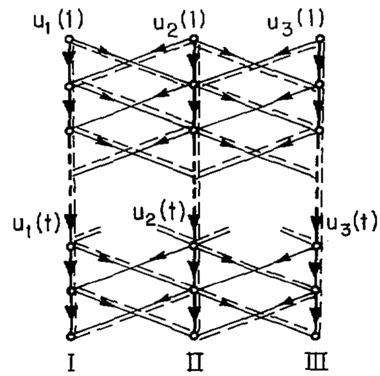


Fig. 6.

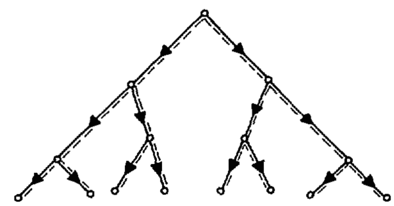


Fig. 7.

Application 2—One-Step Communication Delay Control Systems [6]: Suppose we have two coupled linear-discrete time-dynamic systems controlled by $u_1(t)$ and $u_2(t)$, $t = 1, 2, \cdots, N$, with the usual Gaussian disturbance and noise setup. $y_1(t)$ and $y_2(t)$ are the noise observations for the system. Suppose

$$z_1(t) = \{y_1(\tau), y_2(k) | \tau = 1, 2, \cdots, t; k = 1, \cdots, t-1\}$$
$$z_2(t) = \{y_2(\tau), y_1(k) | \tau = 1, 2, \cdots, t; k = 1, \cdots, t-1\}$$

i.e., two controllers share all past information with one-step communication delay. The information structure diagram of such a system will have the appearance of Fig. 5. It is by inspection partially nested. Hence, if the criterion is quadratic in the state and control variables, then by Theorem 2, the optimal solution is linear. Furthermore, if there is a third linear system coupled to the first system via the second system as shown in Fig. 6, then we may conclude that the first system can tolerate a two-step delay in sharing information with the third system. Since $u_3(t-1)$ does not affect the information $z_1(t)$, we do not have to know $z_3(t-1)$ to maintain linearity of the optimal solution.

Application 3—Hierarchical Control System: Suppose an information structure diagram is that of Fig. 7, which informally represents a chain of commands. Then under linear-quadratic-Gaussian assumptions, the optimal solution is again linear without the need for lateral communication.

Roughly speaking, the implication of Theorem 2, is that, *if a DM's action affects our information, then knowing what he knows will yield linear optimal solutions.*

Application 4—Two-Person Zero-Sum Multistage Games: In the usual formulation of LQG zero-sum decision games, we have

$$x_{i+1} = \Phi x_i + D_1 u_{1i} + D_2 u_{2i} + w_i, \qquad i = 1, \cdots, N \tag{34}$$

$$y_i^1 = H_1 x_i + v_i^1$$
$$y_i^2 = H_2 x_i + v_i^2, \qquad i = 1, \cdots, N \tag{35}$$

$$z_i^1 = \{y_j^1 | j \le i\}$$
$$z_i^2 = \{y_j^2 | j \le i\}, \qquad i = 1, \cdots, N, \tag{36}$$

i.e., each player has perfect memory but does not know the other players' information.

$$J = E\left\{\frac{1}{2} x_{N+1}^T Q_{N+1} x_{N+1} + \sum_{i=1}^{N} \left(x_i^T Q_i x_i + u_{1i}^T R_1 u_{1i} - u_{2i}^T R_2 u_{2i} \right)\right\}. \tag{37}$$

This problem does *not* have partially nested information structure. On the other hand, since the problem is zero sum there is no cooperation feature in the problem. Thus the Problem 1 version of the problem is equivalent to that of Problem 2. Now, if one assumes a linear control law of the type of (20) for all u_{2i}, then $P(z_j^2/z_i^1)$ will be Gaussian for all i, j, and the partial derivative terms in (17) can be explicitly evaluated. The problem from the viewpoint of player one is then an LQG dynamic optimization problem with partially nested information structure. Consequently, u_{1i}^* will have a linear structure of the type of (20). Now, when this control law for u_1 is used again in (17) for u_2, we get the self-consistent result that u_{2i} is linear. These linear saddle point controls are optimal for both players in a global sense by the reason that in a zero-sum game any saddle point strategy will be equivalent and interchangeable [9]. The details of this are best illustrated via a two-stage example where the arithmetic is not too cumbersome.

The point here is that linearity of the optimal solution in an LQG zero-sum multistage game is primarily due to the absence of cooperation and, only as a secondary matter, dependent on the perfect memory feature of the players.

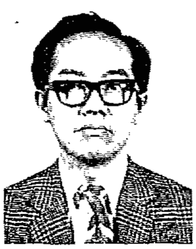
V. Conclusion

In this paper we have essentially answered the question "when does a general linear-quadratic-Gaussian problem have optimal linear solutions" in the context of decentralized multidecision-maker environment. We have shown the importance of the concept of a partially nested information structure diagram which enables the reduction of a dynamic problem of a static one.

References

[1] J. Marschak, "Elements for a theory of teams," *Management Sci.*, vol. 1, pp. 127–137, 1955.
[2] R. Radner, "Team decision problems," *Ann. Math. Statist.*, vol. 33, no. 3, pp. 857–881, 1962.
[3] ——, "The evaluation of information in organizations," in *Proc. 4th Berkeley Symp. Math. Statist. and Probab.* Berkeley, Calif.: University of California Press, 1961, pp. 491–530.
[4] C. Striebel, "Sufficient statistics in the optimum control of stochastic systems," *J. Math. Anal.*, vol. 12, pp. 576–592, 1965.

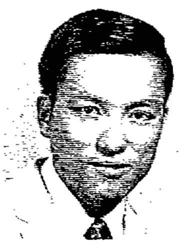
[5] K.-C. Chu and Y.-C. Ho, "On the generalized linear-quadratic-Gaussian problems," in *Differential Games and Related Topics*, H. W. Kuhn and G. P. Szegö, Eds. Amsterdam: North-Holland, 1971, pp. 373–388.
[6] H. S. Witsenhausen, "Separation of estimation and control for discrete time systems," *Proc. IEEE*, vol. 59, pp. 1557–1566, Nov. 1971.
[7] G. Owens, *Game Theory*. Philadelphia, Pa.: Saunders, 1968.
[8] M. DeGroot, *Optimal Statistical Decision Theory*. New York: McGraw-Hill, 1970.
[9] W. W. Willman, "Formal solutions for a class of stochastic pursuit-evasion games," *IEEE Trans. Automat. Contr.*, vol. AC-14, pp. 504–509, Oct. 1969.



Yu-Chi Ho (S'54–M'55–SM'62) was born in China on March 1, 1934. He received the B.S. and M.S. degrees in electrical engineering from the Massachusetts Institute of Technology, Cambridge, in 1953 and 1955, respectively, and the Ph.D. degree in applied mathematics from Harvard University, Cambridge, Mass., in 1961.

From 1955 to 1958 he worked on numerical control systems at the Research Laboratory Division, Bendix Corporation, Mich. Since 1961 he has been on the faculty of Harvard University, where he is presently Gordon McKay Professor of Engineering and Applied Mathematics. He is a consultant to various industrial and research organizations and coinventor of four U.S. patents on various aspects of numerical and digital control systems. He is coauthor of the book *Applied Optimal Control: Optimization, Estimation and Control* (Blaisdell, 1969).

Dr. Ho is a member of the Army Scientific Advisory Panel, a member of the Editorial Board of the IEEE Press, and Associate Editor of the *Journal of Optimization Theory and Applications*. In 1969 he was the Chairman of the First International Conference on the Theory and Application of Differential Games. In 1970 he was a Guggenheim Fellow at Imperial College, London, England, and Cambridge University, Cambridge, England.



K'ai-Ching Chu (S'69–M'71) was born in Szechuan, China, on November 19, 1944. He received the B.S. degree in electrical engineering from the National Taiwan University, Taipei, Taiwan, China, in 1966, and the M.S. and Ph.D. degrees in applied mathematics from Harvard University, Cambridge, Mass., in 1968 and 1971, respectively.

From 1968 to 1971 he was a Research Assistant and Teaching Fellow at Harvard University. He has been a consultant to Bolt, Beranek, and Newman, Inc., Cambridge, Mass. Since September 1971, he has been a Mathematician with Systems Control, Inc., Palo Alto, Calif. His current research interests involve the theory and application of optimal control, statistical decision analysis, and game theory.

Team Decision Theory and Information Structures in Optimal Control Problems—Part II

K'AI-CHING CHU, MEMBER, IEEE

Abstract—**General dynamic team decision problems with linear information structures and quadratic payoff functions are studied. The primitive random variables are jointly Gaussian. No constraints on the information structures are imposed except causality.**

Equivalence relations in information and in control functions among different systems are developed. These equivalence relations aid in the solving of many general problems by relating their solutions to those of the systems with "perfect memory." The latter can be obtained by the method derived in Part I. A condition is found which enables each decision maker to infer the information available to his precedents, while at the same time the controls which will affect the information assessed can be proven optimal. When this condition fails, upper and lower bounds of the payoff function can still be obtained systematically, and suboptimal controls can be obtained.

Manuscript received April 19, 1971. Paper recommended by P. P. Varaiya, past Chairman of the IEEE S-CS Large Systems, Differential Games Committee. This research was supported by the Joint Services Electronics Program under Contract N00014-67-A-0298-0006 through Harvard University.

The author was with the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass. He is now with Systems Control, Inc., Palo Alto, Calif. 94306.

I. Introduction

In Part I of this paper, Ho and Chu [1] have discussed the information structures in a general organization and their relation to team decision problems. It is found that in a general causal system a partially ordered precedence relation $\{$ can be defined among all the members. This precedence relation then specifies the nature of the solution.

A linear-quadratic-Gaussian (LQG) team problem (Q, S, c, H_i, $D_{ij} | i, j = 1, \cdots, N$) is an optimal decision problem with payoff function

$$J = E[\mathcal{J}] = E[\tfrac{1}{2}\boldsymbol{u}^TQ\boldsymbol{u} + \boldsymbol{u}^TS\xi + \boldsymbol{u}^Tc] \qquad (1)$$

where $\boldsymbol{u}^T = (u_1{}^T, \cdots, u_N{}^T)$ and u_i is the action variable of team member i; matrices Q, S and vector c are fixed and of appropriate dimensions, Q is symmetric positive definite; the random variable of the external world ξ is *a priori* Gaussian with distribution $N(\mathbf{O}, X)$. The information z_i