On Computation of Optimal Controllers Subject to Quadratically Invariant Sparsity Constraints

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Abstract

We consider the problem of constructing optimal sparse controllers. It is known that a property called quadratic invariance of the constraint set is important, and results in the constrained minimum-norm problem being soluble via convex programming. We provide an explicit method of computing $\mathcal{H}_2$-optimal controllers subject to quadratically invariant sparsity constraints, along with a computational test for quadratic invariance. As a consequence, we show that block diagonal constraints are never quadratically invariant unless the plant is block diagonal as well.

Keywords: Decentralized control, convex optimization

1 Introduction

An important problem in control is that of constructing decentralized control systems, where instead of a single controller connected to a physical system, one has multiple controllers, each with access to different information. Examples of such systems include the electricity distribution grid, automobiles on the freeway, flocks of aerial vehicles, paper machining, and spacecraft moving in formation.

In a standard controls framework, the decentralization of the system manifests itself as sparsity or delay constraints on the controller to be designed. These constraints vary depending on the structure of the physical systems, and how separate controllers can communicate. In general, there is no known method of formulating the problem of finding a norm-minimizing controller subject to such constraints as a convex optimization problem. In many cases the problem is intractable.

It has been shown that if the constraints on the controller satisfy a particular property, called *quadratic invariance*, with respect to the system being controlled, then the constrained minimum-norm control problem may be reduced to a convex optimization problem. Such quadratically invariant constraints arise in many practical contexts. In this paper we provide a computational test for quadratic invariance when the controller structure is defined by sparsity constraints. We further give a procedure for computing the $\mathcal{H}_2$-optimal controller subject to quadratically invariant sparsity constraints.

1.1 Prior work

The research in this area has a long history, and there have been many striking results which illustrate the complexity of this problem. Important early work includes that of Radner [9], who developed sufficient conditions under which minimal quadratic cost for a linear system is achieved by a linear controller. An important example was presented in 1968 by Witsenhausen [14] where it was shown that for quadratic stochastic optimal control of a linear system, subject to a decentralized information constraint called *non-classical information*, a nonlinear controller can achieve greater performance than any linear controller. An additional consequence of the work of [6, 14] is to show that under such a non-classical information pattern the cost function is no longer convex in the controller variables, a fact which today has increasing importance.

With the difficulty of the general problem elucidated, efforts followed to classify when linear controllers were indeed optimal, when finding the optimal linear controller could be cast as a convex optimization problem, and to understand the complexity of decentralized control problems. In a later paper [15], Witsenhausen summarized several important results on decentralized control at that time, and gave sufficient conditions under which the problem could be reformulated so that the standard Linear-Quadratic-Gaussian (LQG) theory could be applied. Under these conditions, an optimal decentralized controller for a linear system could be chosen to be linear. Ho and Chu [4], in the framework of *team theory*, defined a more general class of information structures, called *partially nested*, for which they showed the optimal LQG controller to be linear. Roughly speaking, a plant-controller system is called partially nested if whenever the information of con-

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troller A is affected by the decision of a controller B, then A has access to all information that B has.

Certain decentralized control problems, such as the static team problem of [9], have been proven to be intractable. Blondel and Tsitsiklis [2] showed that the problem of finding a stabilizing decentralized static output feedback is NP-complete. This is also the case for a discrete variant of Witsenhausen's counterexample [7].

For particular information structures, the controller optimization problem may have a tractable solution, and in particular, it was shown by Voulgaris [12] that the so-called *one-step delay information sharing pattern* problem has this property. In [3] the LEQG problem is solved in this framework, and in [12] the $\mathcal{H}_2$, $\mathcal{H}_\infty$ and $\mathcal{L}_1$ control synthesis problems are solved. A class of structured space-time systems has also been analyzed in [1], and shown to be reducible to a convex program.

It was shown in [10] that a property called *quadratic invariance* is necessary and sufficient for the constraint set to be preserved under feedback. This allows the constrained minimum-norm control problem to be reduced to a convex optimization problem, and the tractable structures of [1, 3, 4, 8, 12, 13, 15] can all be shown to satisfy this property. In this paper we focus on the case when the subspace is defined by sparsity constraints, and provide an explicit method of computing the $\mathcal{H}_2$-optimal sparse controller subject to such constraints, along with a computational test for quadratic invariance.

1.2 Preliminaries

We use the following standard notation. Denote by $\mathcal{R}_p^{m\times n}$ the set of matrix-valued real-rational proper transfer matrices and let $\mathcal{R}_{sp}^{m\times n}$ be the set of real-rational strictly proper transfer matrices. We will omit the spatial dimensions when they are clear from context.

Suppose $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, and partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where $P_{11} \in \mathcal{R}_p^{n_z\times n_w}$. For $K \in \mathcal{R}_p^{n_u\times n_y}$ such that $I - P_{22}K$ is invertible, the ***linear fractional transformation*** (LFT) of P and K is denoted $f(P,K)$, and is defined by

$$f(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

In the remainder of the paper, we abbreviate our notation and define $G = P_{22}$. This interconnection is shown in Figure 1. We will also refer to $f(P,K)$ as the ***closed-loop map***.

Given $A \in \mathbb{C}^{m\times n}$ associate a vector $\text{vec}(A) \in \mathbb{C}^{mn}$ defined by

$$\text{vec}(A) = \begin{bmatrix} A_{11} \; ... \; A_{m1} \; A_{12} \; ... \; A_{m2} \; ..... \; A_{1n} \; ... \; A_{mn} \end{bmatrix}^{\text{T}}$$

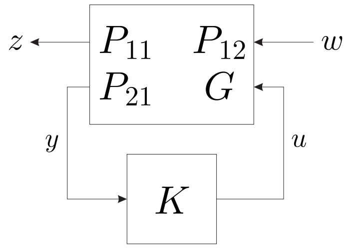

Figure 1: Linear fractional interconnection of P and K

Given $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{s\times q}$ let $A \otimes B \in \mathbb{C}^{ms\times nq}$ denote the *Kronecker product* of A and B.

Lemma 1. *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{s\times q}, X \in \mathbb{C}^{n\times s}$. Then*

$$vec(AXB) = (B^T \otimes A)vec(X)$$

Proof. See, for example, [5]. ■

Definition 2. *We define the map $h : \mathcal{R}_p \times \mathcal{R}_p \to \mathcal{R}_p$ by*

$$h(G,K) = -K(I - GK)^{-1} \quad \text{for all } G, K \text{ such that } I - GK \text{ is invertible}$$

1.3 Problem Formulation

Suppose $S \subset \mathcal{R}_p^{n_u\times n_y}$ is a subspace. Given $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, we would like to solve the following problem

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned} \tag{1}$$

Here $\|\cdot\|$ is any norm on $\mathcal{R}_p^{n_z\times n_w}$, chosen to encapsulate the control performance objectives, and S is a subspace of admissible controllers which characterizes the decentralized nature of the system. We call the subspace S the ***information constraint***.

Many decentralized control problems may be expressed in this form. In this paper, we focus on the case where the norm on $\mathcal{R}_p^{n_z\times n_w}$ is the $\mathcal{H}_2$-norm, and where S is defined by sparsity constraints.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved with many standard techniques. Note that the cost function $\|f(P,K)\|$ is in general a non-convex function of K. No computationally tractable approach is known for solving this problem for arbitrary P and S.

1.4 Example

Many standard centralized and decentralized control problems may be represented in the form of problem (1), for specific choices of P and S. The following is an example.

Perfectly decentralized control. We would like to design n separate controllers $K_1, \dots, K_n$, with controller K_i connected to subsystem G_i of a coupled system, as in Figure 2. When reformulated as a synthesis problem in the LFT form above, the constraint set S is

$$S = \left\{ K \in \mathcal{R}_p^{n_u \times n_y} \mid K = \operatorname{diag}(K_1, \dots, K_n) \right\}$$

that is, S consists of those controllers that are *block-diagonal*.

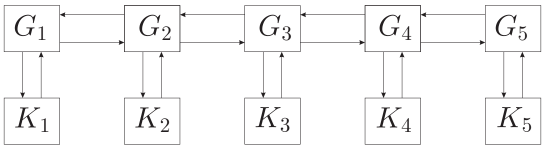

Figure 2: Perfectly decentralized control

We return to this example in Section 2.

1.5 Quadratic Invariance

In [10], a property called *quadratic invariance* was introduced for general linear operators. We define this here for the special case of transfer functions.

Definition 3. *Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$, and $S \subset \mathcal{R}_p^{n_u \times n_y}$. The set S is called **quadratically invariant** under G if*

$$KGK \in S \qquad \text{for all } K \in S$$

Note that, given G, we can define a quadratic map $\Psi : \mathcal{R}_p^{n_u \times n_y} \to \mathcal{R}_p^{n_u \times n_y}$ by $\Psi(K) = KGK$. Then a set S is quadratically invariant if and only if S is an invariant set of Ψ; that is $\Psi(S) \subset S$.

It was shown in [10, 11] that this condition allows (1) to be formulated as a convex optimization problem. In this paper, we provide a computational test for this condition when the constraint set is defined by sparsity constraints, and then give a procedure for solving the resulting convex optimization problem.

2 Computational Test

Many problems in decentralized control can be expressed in the form of problem (1), where S is the set of controllers that satisfy a specified sparsity constraint. In this section, we provide a computational test for quadratic invariance when the subspace S is defined by block sparsity constraints. First we introduce some notation.

Suppose $A^{\text{bin}} \in \{0,1\}^{m \times n}$ is a binary matrix. We define the subspace

$$\operatorname{Sparse}(A^{\text{bin}}) = \Big\{ B \in \mathcal{R}_p \mid B_{ij}(j\omega) = 0 \text{ for all } i,j \\ \text{such that } A_{ij}^{\text{bin}} = 0 \text{ for almost all } \omega \in \mathbb{R} \Big\}$$

Also, if $B \in \mathcal{R}_{sp}$, let $A^{\text{bin}} = \operatorname{Pattern}(B)$ be the binary matrix given by

$$A_{ij}^{\text{bin}} = \begin{cases} 0 & \text{if } B_{ij}(j\omega) = 0 \text{ for almost all } \omega \in \mathbb{R} \\ 1 & \text{otherwise} \end{cases}$$

Note that in this section, we assume that matrices of transfer functions are indexed by blocks, so that above, the dimensions of A^{bin} may be much smaller than those of B. Then, K_{kl}^{bin} determines whether controller k may use measurements from subsystem l, K_{kl} is the map from the outputs of subsystem l to the inputs of subsystem k, and G_{ij} represents the map from the inputs to subsystem j to the outputs of subsystem i.

The following results were presented for scalar sparsity constraints in [11]. We provide an extension to block sparsity constraints. We first prove two preliminary lemmas.

Lemma 4. *Suppose $S = \operatorname{Sparse}(K^{bin})$, and let $G^{bin} = \operatorname{Pattern}(G)$. If S is quadratically invariant under G, then*

$$K_{ki} = 0 \text{ or } K_{jl} = 0 \text{ for all } (i,j,k,l) \text{ and } K \\ \text{such that } K_{kl}^{bin} = 0,\ G_{ij}^{bin} = 1,\ K \in S$$

Proof. Suppose there exists (i,j,k,l) and K such that

$$K_{kl}^{\text{bin}} = 0,\ G_{ij}^{\text{bin}} = 1,\ K \in S$$

but

$$K_{ki} \neq 0 \text{ and } K_{jl} \neq 0$$

Then we must have

$$K_{ki}^{\text{bin}} = 1,\ K_{jl}^{\text{bin}} = 1,\ i \neq l,\ j \neq k$$

Consider $K \in S$ such that

$$K_{ab} = 0 \text{ if } (a,b) \notin \{(k,i),(j,l)\}$$

Then

$$(KGK)_{kl} = \sum_r \sum_s K_{kr} G_{rs} K_{sl} = K_{ki} G_{ij} K_{jl}$$

Since $G_{ij} \neq 0$, we can easily choose K_{ki} and K_{jl} such that $(KGK)_{kl} \neq 0$. So $KGK \notin S$ and S is not quadratically invariant.

■

Lemma 5. *Suppose $S = \operatorname{Sparse}(K^{bin})$, and let $G^{bin} = \operatorname{Pattern}(G)$. If*

$$K_{ki} = 0 \text{ or } K_{jl} = 0 \text{ for all } (i,j,k,l) \text{ and } K \\ \text{such that } K_{kl}^{bin} = 0,\ G_{ij}^{bin} = 1,\ K \in S$$

Then

$$K_{ki}^{bin} K_{jl}^{bin} = 0 \text{ for all } (i,j,k,l) \\ \text{such that } K_{kl}^{bin} = 0,\ G_{ij}^{bin} = 1$$

Proof. We show this by contradiction. Suppose there exists (i,j,k,l) such that

$$K_{kl}^{\text{bin}} = 0, \quad G_{ij}^{\text{bin}} = 1, \quad K_{ki}^{\text{bin}} K_{jl}^{\text{bin}} \neq 0.$$

Then

$$K_{ki}^{\text{bin}} \;=\; K_{jl}^{\text{bin}} \;=\; 1$$

and hence it must follow that there exists $K \in S$ such that $K_{ki} \neq 0$ and $K_{jl} \neq 0$. ∎

The following is the main result of this section. It provides a computational test for quadratic invariance when S is defined by sparsity constraints. It also equates quadratic invariance with a stronger condition.

Theorem 6. *Suppose $S = \text{Sparse}(K^{bin})$, and let $G^{bin} = \text{Pattern}(G)$. Then the following are equivalent:*

(i) S is quadratically invariant under G

(ii) $KGJ \in S$ for all $K, J \in S$

(iii) $K_{ki}^{bin}\ G_{ij}^{bin}\ K_{jl}^{bin}\ (1 - K_{kl}^{bin}) = 0$ for all $i, l = 1, \ldots, n_y$ and $j, k = 1, \ldots, n_u$

Proof. We will show that $(i) \implies (iii) \implies (ii) \implies (i)$. Suppose S is quadratically invariant under G. Then by Lemma 4,

$$\begin{aligned} &K_{ki} = 0 \text{ or } K_{jl} = 0 \text{ for all } (i,j,k,l) \text{ and } K \\ &\qquad\qquad \text{such that } K_{kl}^{\text{bin}} = 0;\ G_{ij}^{\text{bin}} = 1;\ K \in S \end{aligned}$$

and by Lemma 5,

$$\begin{aligned} &K_{ki}^{\text{bin}} K_{jl}^{\text{bin}} = 0 \text{ for all } (i,j,k,l) \\ &\qquad\qquad\qquad \text{such that } K_{kl}^{\text{bin}} = 0,\ G_{ij}^{\text{bin}} = 1 \end{aligned}$$

which can be restated

$$K_{ki}^{\text{bin}}\ G_{ij}^{\text{bin}}\ K_{jl}^{\text{bin}}\ (1 - K_{kl}^{\text{bin}}) \;=\; 0$$

and which implies that

$$\begin{aligned} &K_{ki} = 0 \text{ or } J_{jl} = 0 \text{ for all } (i,j,k,l),\ K,\ J \\ &\qquad\qquad \text{such that } K_{kl}^{\text{bin}} = 0;\ G_{ij}^{\text{bin}} = 1;\ K, J \in S \end{aligned}$$

which clearly implies

$$\begin{aligned} &(KGJ)_{kl} \;=\; \sum_i \sum_j K_{ki} G_{ij} J_{jl} \;=\; 0 \\ &\qquad \text{for all } (k,l), K, J \text{ such that } K_{kl}^{\text{bin}} = 0; K, J \in S \end{aligned}$$

and thus

$$KGJ \in S \text{ for all } K, J \in S$$

which is a stronger condition than quadratic invariance and hence implies (i). ∎

This result shows us several things about sparsity constraints. In this case quadratic invariance is equivalent to another condition which is stronger in general. When G is symmetric, for example, the subspace consisting of symmetric K is quadratically invariant but does not satisfy condition (ii). Condition (iii), which gives us the computational test we desired, shows that quadratic invariance can be checked in time $O(n^4)$, where $n = \max\{n_u, n_y\}$. It also shows that, if S is defined by sparsity constraints, then S is quadratically invariant under G if and only if it is quadratically invariant under all systems with the same sparsity pattern.

Perfectly Decentralized Control. We now show an interesting negative result. Let $n_u = n_y$, so that each subsystem has its own controller as in Figure 2.

Corollary 7. *Suppose there exists i, j, with $i \neq j$, such that $G_{ij} \neq 0$. Suppose K^{bin} is diagonal and $S = \text{Sparse}(K^{\text{bin}})$. Then S is not quadratically invariant under G.*

Proof. Let $G^{\text{bin}} = \text{Pattern}(G)$. Then

$$K_{ii}^{\text{bin}}\ G_{ij}^{\text{bin}}\ K_{jj}^{\text{bin}}\ (1 - K_{ij}^{\text{bin}}) = 1$$

The result then follows from Theorem 6. ∎

It is important to note that the plant and controller do not have to be square to apply this result because of the block notation used in this section. This corollary tells us that if each subsystem has its own controller which may only use sensor information from its own subsystem, and any subsystem affects any other, then the system *is not* quadratically invariant. In other words, perfectly decentralized control is never quadratically invariant except for the trivial case where no subsystem affects any other.

3 Computation of Optimal Controllers

Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ and $S \subset \mathcal{R}_p^{n_u \times n_y}$ is a subspace defined by sparsity constraints. We would like to solve problem (1). It was shown in [11] that if S is quadratically invariant under G and $K_{\text{nom}} \in \mathcal{RH}_\infty \cap S$ is a stabilizing controller, then K is optimal for this problem if and only if $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$ and Q is optimal for

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & Q \in S \end{aligned} \tag{2}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$ are given by

$$\begin{aligned} T_1 &= P_{11} + P_{12} K_{\text{nom}} (I - G K_{\text{nom}})^{-1} P_{21} \\ T_2 &= -P_{12} (I - K_{\text{nom}} G)^{-1} \\ T_3 &= (I - G K_{\text{nom}})^{-1} P_{21} \end{aligned} \tag{3}$$

This result holds for any norm. We show in this section that if we wish to minimize the $\mathcal{H}_2$-norm, then we can convert (2) to an unconstrained problem which may be readily solved.

For ease of presentation, we now make a slight change of notation from the previous section. We no longer assume that the plant and controller are divided into blocks, so that K^{bin}_{kl} now determines whether the kl index of the controller may be non-zero, rather than determining whether controller k may use information from subsystem l, and G_{ij} similarly represents the ij index of the plant. K^{bin} therefore has the same dimension as the controller itself. n_u and n_y represent the total number of inputs and outputs, respectively.

Let

$$a = \sum_{i=1}^{n_u} \sum_{j=1}^{n_y} K^{\text{bin}}_{ij}$$

such that a represents the number of admissible controls, that is, the number of indices for which K is not constrained to be zero.

The following theorem gives the equivalent unconstrained problem.

Theorem 8. *Suppose x is an optimal solution to*

$$\begin{aligned} &\textit{minimize} && \|b + Ax\| \\ &\textit{subject to} && x \in \mathcal{RH}_\infty \end{aligned} \tag{4}$$

where $D \in \mathbb{R}^{n_u n_y \times a}$ is a matrix whose columns form an orthonormal basis for $vec(S)$, and

$$b = vec(T_1), \qquad A = -(T_3^T \otimes T_2)D.$$

Then $Q = vec^{-1}(Dx)$ is optimal for (2) and the optimal values are equivalent.

Proof. We know that

$$Q \in \mathcal{RH}_\infty^{n_u \times n_y} \cap S \iff \begin{array}{l} \text{vec}(Q) = Dx \\ \text{for some } x \in \mathcal{RH}_\infty^{a \times 1} \end{array}$$

Since

$$\begin{aligned} &\|T_1 - T_2 Q T_3\|_2 \\ &= \|\text{vec}(T_1 - T_2 Q T_3)\|_2 \quad \text{by definition of the } \mathcal{H}_2\text{-norm} \\ &= \|\text{vec}(T_1) - (T_3^{\text{T}} \otimes T_2)\text{vec}(Q)\|_2 \quad \text{by Lemma 1} \\ &= \|\text{vec}(T_1) - (T_3^{\text{T}} \otimes T_2)Dx\|_2 \\ &= \|b + Ax\|_2 \end{aligned}$$

we have the desired result. ∎

Therefore, we can find the optimal x for problem (4) using many available tools for unconstrained $\mathcal{H}_2$-synthesis, with

$$P_{11} = b \qquad P_{12} = A \qquad P_{21} = 1 \qquad P_{22} = 0^{1 \times a}$$

then find the optimal Q for problem (2) as $Q = \text{vec}^{-1}(Dx)$, and finally, find the optimal K for problem (1) as $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$.

4 Numerical Example

Consider an unstable lower triangular plant

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s-1} \end{bmatrix}$$

with P given by

$$P_{11} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \qquad P_{12} = \begin{bmatrix} G \\ I \end{bmatrix} \qquad P_{21} = \begin{bmatrix} G & I \end{bmatrix}$$

and a sequence of sparsity constraints $K^{\text{bin}}_1, \ldots, K^{\text{bin}}_6$

$$K^{\text{bin}}_1 = \begin{bmatrix} 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&1 \end{bmatrix} \quad K^{\text{bin}}_2 = \begin{bmatrix} 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&0 \\ 1&1&0&0&1 \end{bmatrix}$$

$$K^{\text{bin}}_3 = \begin{bmatrix} 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&0 \\ 1&1&0&0&0 \\ 1&1&0&0&1 \end{bmatrix} \quad K^{\text{bin}}_4 = \begin{bmatrix} 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&0 \\ 1&1&0&0&0 \\ 1&1&1&0&1 \end{bmatrix}$$

$$K^{\text{bin}}_5 = \begin{bmatrix} 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&0 \\ 1&1&1&0&0 \\ 1&1&1&0&1 \end{bmatrix} \quad K^{\text{bin}}_6 = \begin{bmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 1&1&1&0&0 \\ 1&1&1&1&0 \\ 1&1&1&1&1 \end{bmatrix}$$

defining a sequence of information constraints $S_i = \text{Sparse}(K^{\text{bin}}_i)$ such that each subsequent constraint is less restrictive, and such that each is quadratically invariant under G. We also use S_7 as the set of controllers with no sparsity constraints; i.e., the centralized case. A stable and stabilizing controller which lies in the subspace defined by any of these sparsity constraints is given by

$$K_{\text{nom}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-6}{s+3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-6}{s+3} \end{bmatrix}$$

We can then find T_1, T_2, T_3 as in (3), and then find the stabilizing controller which minimizes the closed-loop norm subject to the sparsity constraints by solving problem (4), as outlined in Section 3. The graph in Figure 3 shows the resulting minimum $\mathcal{H}_2$ norms for the six sparsity constraints as well as for a centralized controller.

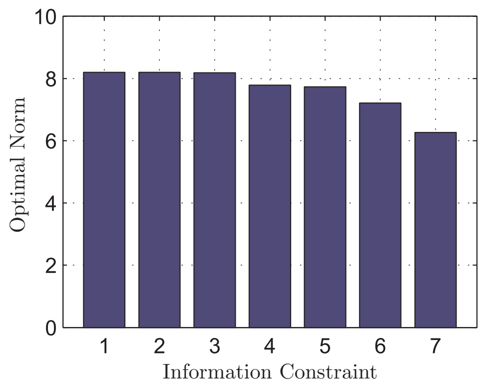

Figure 3: Optimal norm with information constraints

5 Conclusion

We gave a computational test for quadratic invariance when the controller is subject to sparsity constraints. As a corollary, we noted that synthesis of block diagonal controllers is only quadratically invariant if the plant is block diagonal as well. We then provided an explicit computational method for synthesizing $\mathcal{H}_2$-optimal controllers subject to quadratically invariant sparsity constraints.

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