Information Structures Preserved Under Nonlinear Time-Varying Feedback

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Abstract

The problem of parameterizing all stabilizing controllers subject to constraints is considered. A new condition relating the plant and the constraints is introduced, and it is shown that when it holds, the constraints are invariant under feedback.

There is no assumption of linearity or time-invariance on either the plant or the admissible controllers, only causality. There is also no assumption that the constraint set be a subspace, only that it is closed. The results hold in continuous time or discrete time.

When the condition holds, we then obtain a parameterization of all stabilizing controllers subject to the specified constraints, which is a convex parameterization as long as the desired constraint set is convex.

1 Introduction

The problem of parameterizing all stabilizing controllers subject to constraints is considered. A new condition relating the plant and the constraints is introduced, and it is shown that when it holds, the constraints are invariant under feedback.

A similar result was achieved for a conditioned called quadratic invariance [7], which assumed that both the plant and all of the controllers under consideration were linear and time-invariant, and also assumed that the set of admissible controllers was a subspace, as is associated with decentralized control.

In this paper, there is no assumption of linearity or time-invariance on either the plant or the admissible controllers, only causality. There is also no assumption that the constraint set be a subspace, only that it is closed. Because of this, these invariance results are no longer only applicable to decentralized control problems, but to other types of constrained control as well.

When the condition holds, we then obtain a parameterization of all stabilizing controllers subject to the specified constraints, which is a convex parameterization as long as the desired constraint set is convex.

1.1 Overview

In Section 2 we introduce some of the notation and terminology that will be used throughout the paper. In Section 3, we prove that the feedback map may be obtained via iteration, and provide conditions under which this is so. Section 4 provides the main result, which introduces our new condition and then shows the constraint set to be invariant under the feedback map. Section 5 then uses this to parameterize all stabilizing constrained controllers. We make some concluding remarks in Section 6.

2 Preliminaries

Throughout this paper, there are many statements, including the main result, which can be made both for continuous time and for discrete time. Rather than state each of these twice, we use $\mathcal{T}_+$ to refer to both $\mathbb{R}_+$ and $\mathbb{Z}_+$, and then use $\mathcal{L}_e$ to refer to the corresponding extended space, as defined below.

We introduce some notation for extended spaces. These spaces are utilized extensively in [5, 8].

We define the truncation operator P_T for all $T \in \mathcal{T}_+$ on all functions $f : \mathcal{T}_+ \to \mathbb{R}$ such that $f_T = P_T f$ is given by

$$f_T(t) = \begin{cases} f(t) & \text{if } t \leq T \\ 0 & \text{if } t > T \end{cases}$$

We define another projection $\tilde{P}_{t,\Delta}$ for all $t, \Delta \in \mathbb{R}$ as

$$\tilde{P}_{t,\Delta} \;=\; P_{t+\Delta} - P_t$$

Note that in discrete time, we would just have

$$\tilde{P}_{k,1}x \;=\; x_{k+1}$$

and we only utilize the latter notation.

For continuous time, we make use of the standard L_p Banach spaces

$$L_p = \left\{ f : \mathbb{R}_+ \to \mathbb{R} \;\middle|\; \int_0^\infty (f(t))^p \text{ exists and is finite} \right\}$$

equipped with the usual p-norm

$$\|f\|_p = \left(\int_0^\infty (f(t))^p \, dt \right)^{\frac{1}{p}}$$

for any $p \geq 1$, and the extended spaces

$$L_{pe} = \{ f : \mathbb{R}_+ \to \mathbb{R} \mid P_T f \in L_p \; \forall \, T \in \mathbb{R}_+ \}$$

We similarly extend the discrete time Banach spaces ℓ_p to the extended space

$$\ell_e = \{ f : \mathbb{Z}_+ \to \mathbb{R} \mid f_T \in \ell_\infty \; \forall \, T \in \mathbb{Z}_+ \}$$

Note that in discrete time, all extended spaces contain the same elements, since the common requirement is that the sequence is finite at any finite index. This motivates the abbreviated notation of ℓ_e.

We let the topology on L_{pe} be generated by the sufficient family of semi-norms $\{\|\cdot\|_T \mid T \in \mathbb{R}_+\}$ where $\|f\|_T = \|P_T f\|_{L_p}$. We let the topology on ℓ_e be generated by the sufficient family of semi-norms $\{\|\cdot\|_T \mid T \in \mathbb{Z}_+\}$ where $\|f\|_T = \|P_T f\|_{\ell_2}$. Thus convergence of a sequence on every possible truncation $P_T \mathcal{L}_e$ implies convergence on $\mathcal{L}_e$.

When the dimensions are implied by context, we omit the superscripts of $\mathcal{L}_e^{m \times n}, L_{pe}^{m \times n}, \ell_e^{m \times n}$.

An operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is said to be causal iff

$$P_T H P_T \;=\; P_T H \qquad \forall \, T \in \mathcal{T}_+$$

that is, if and only if future inputs can't affect past or present outputs.

A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is said to be finite gain stable, hereafter abbreviated as f.g. stable, iff there exists $\gamma < \infty$ such that $\forall \, T \in \mathcal{T}_+$ and $\forall \, x \in \mathcal{L}_e$

$$\|P_T H x\| \;\leq\; \gamma \|P_T x\|$$

A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is said to be incrementally stable, hereafter abbreviated as inc. stable, iff there exists $\gamma < \infty$ such that $\forall \, T \in \mathcal{T}_+$ and $\forall \, x, y \in \mathcal{L}_e$

$$\|P_T H x - P_T H y\| \;\leq\; \gamma \|P_T x - P_T y\|$$

We say that $K : \mathcal{L}_e \to \mathcal{L}_e$ f.g. stabilizes $G : \mathcal{L}_e \to \mathcal{L}_e$ iff for the interconnection in Figure 1 the maps from the two inputs to the four other signals are all f.g. stable.

We say that $G : \mathcal{L}_e \to \mathcal{L}_e$ is strongly stabilizable iff there exists $K : \mathcal{L}_e \to \mathcal{L}_e$ which is inc. stable and which f.g. stabilizes G.

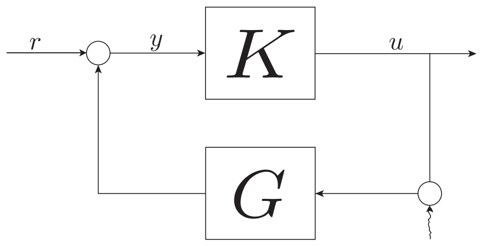

Figure 1: Interconnection of K and G

3 Iteration

In this section, we prove that the feedback map from r to u in Figure 1 may be obtained via iteration, and provide conditions under which this is so. In Section 3.1 we give very broad assumptions under which all the signals of interest are guaranteed to be uniquely defined. In Section 3.2 we introduce the iterations for both signals and operators which will be used throughout the paper, and for which we have to prove convergence. In Section 3.3 we introduce an additional continuity assumption and prove convergence of the signals, and in Section 3.4 we prove convergence of the operators.

This section is extremely technical. The reader wishing to skip ahead should just take away that an iteration $R^{(n)}$ is introduced which will be used in proving the main result, and that it converges to the map from r to u in the figure. Technical conditions are developed which guarantee this convergence, they include all operators which are strictly causal and continuous, and they will be assumed to hold throughout the remainder of the paper.

3.1 Existence and uniqueness

We wish to establish broad conditions under which equations of the form $r = e - He$ are guaranteed to have a unique solution. For example, in Figure 1 we may wish to know if all of the signals are uniquely defined, and would then seek solutions with

$$H = \begin{bmatrix} 0 & G \\ K & 0 \end{bmatrix}$$

We now state a theorem which gives conditions under which a unique solution is guaranteed to exist.

Theorem 1. *Suppose $H : L_{1e} \to L_{1e}$ is causal. If for all compact intervals $I \subseteq \mathbb{R}_+$, there exists $\gamma(I) < 1$ and $\Delta(I) > 0$ such that $\forall \, t \in I$ and $\forall \, e, e' \in L_{1e}$ subject to $P_t e = P_t e'$,*

$$\|\tilde{P}_{t,\Delta(I)}(He - He')\| \;\leq\; \gamma(I) \; \|\tilde{P}_{t,\Delta(I)}(e - e')\|$$

then for any $r \in L_{1e}$, the equation $r = e - He$ has one and only one solution.

Proof. See, for example, Section III.5 of [5]. ■

We can state an analogous result for the discrete-time case, and provide a proof in the same spirit.

Theorem 2. *Suppose $H : \ell_e \to \ell_e$ is causal. If for all $k \in \mathbb{Z}_+$ there exists $\gamma_k < 1$ such that $\forall\, e, e' \in \ell_e$ subject to $P_k e = P_k e'$,*

$$\|(He - He')_{k+1}\| \;\leq\; \gamma_{k+1}\, \|(e - e')_{k+1}\|$$

then for any $r \in \ell_e$, the equation $r = e - He$ has one and only one solution.

Proof. Suppose that we have a solution for e up to some index $k \in \mathbb{Z}_+$. We use the causality of H, and define a function on the subsequent value as follows

$$\begin{aligned} e_{k+1} &= r_{k+1} + (H(P_k e + e_{k+1}))_{k+1} \\ &= f_{k+1}(e_{k+1}) \end{aligned}$$

Now if we are given $e_{k+1}, \tilde{e}_{k+1} \in \mathbb{R}$

$$\begin{aligned} &\|f_{k+1}(e_{k+1}) - f_{k+1}(\tilde{e}_{k+1})\| \\ &= \left\|(H(P_k e + e_{k+1}))_{k+1} - (H(P_k e + \tilde{e}_{k+1}))_{k+1}\right\| \\ &\leq \gamma\, \|e_{k+1} - \tilde{e}_{k+1}\| \end{aligned}$$

Thus f_{k+1} is a contraction on $\mathbb{R}$, and so the iteration

$$e_{k+1}^{(n+1)} = f_{k+1}\big(e_{k+1}^{(n)}\big) \qquad n = 0, 1, 2, \ldots$$

will converge to the unique solution of e_{k+1} for any initial value $e_{k+1}^{(0)} \in \mathbb{R}$. Thus by starting at $k = 0$ and repeating this process for every $k \in \mathbb{Z}_+$, we can uniquely construct the solution for $e \in \ell_e$. ■

These conditions basically say that if you look over a short enough horizon, the map needs to be contractive. For discrete time systems, that is always achieved with strict causality, and a small enough component which is not strictly causal may be present as well. For finite-dimensional linear time-invariant systems, continuous or discrete, this would simply translate to being strictly proper, or at least requiring the feed-through term to have norm less than one. For more general continuous time systems, we get the condition above.

3.2 Closed-loop iterations

Given $K : \mathcal{L}_e \to \mathcal{L}_e$ and $G : \mathcal{L}_e \to \mathcal{L}_e$ define $R^{(n)} : \mathcal{L}_e \to \mathcal{L}_e$ and $Y^{(n)} : \mathcal{L}_e \to \mathcal{L}_e$ for each $n \in \mathbb{Z}_+$ as

$$\begin{aligned} Y^{(0)} &= I \\ R^{(n)} &= KY^{(n)} && \forall\, n \in \mathbb{Z}_+ \\ Y^{(n+1)} &= I + GR^{(n)} && \forall\, n \in \mathbb{Z}_+ \end{aligned}$$

so then $Y^{(n)}$ may be defined recursively as

$$\begin{aligned} Y^{(0)} &= I \\ Y^{(n+1)} &= I + GKY^{(n)} && \forall\, n \in \mathbb{Z}_+ \end{aligned}$$

and $R^{(n)}$ may be defined recursively as

$$\begin{aligned} R^{(0)} &= K \\ R^{(n+1)} &= K(I + GR^{(n)}) && \forall\, n \in \mathbb{Z}_+ \end{aligned}$$

Now consider an arbitrary $r \in \mathcal{L}_e$, and define $y^{(n)} \in \mathcal{L}_e$ and $u^{(n)} \in \mathcal{L}_e$ for each $n \in \mathbb{Z}_+$ as

$$y^{(n)} = Y^{(n)}(r), \qquad u^{(n)} = R^{(n)}(r) \qquad \forall\, n \in \mathbb{Z}_+$$

We then get the following iteration, commensurate with the block diagram

$$\begin{aligned} y^{(0)} &= r \\ u^{(n)} &= Ky^{(n)} \\ y^{(n+1)} &= r + Gu^{(n)} \end{aligned}$$

Then $y^{(n)}$ may be defined recursively as

$$\begin{aligned} y^{(0)} &= r \\ y^{(n+1)} &= r + GKy^{(n)} \end{aligned}$$

and $u^{(n)}$ may be defined recursively as

$$\begin{aligned} u^{(0)} &= Kr \\ u^{(n+1)} &= K(r + Gu^{(n)}) \end{aligned}$$

3.3 Convergence of signals

We now wish to consider the convergence of iterations such as those for $y^{(n)}$ above. This is more difficult than showing the convergence of the iterations in the existence and uniqueness proofs of Theorem 1 or Theorem 2, because we don't have the luxury of allowing complete convergence at one index or time interval before moving on to the next. In other words, we must show that, at each time interval, the iteration still converges, even though $P_t y \neq P_t y'$ over previous intervals but is arbitrarily close.

We need to introduce a continuity condition for this.

Suppose that for every $T \in \mathcal{T}_+$ there exists $\beta_T < \infty$ such that for all $x, y \in \mathcal{L}_e$

$$\|P_T Hx - P_T Hy\| \;<\; \beta_T \|P_T x - P_T y\| \tag{1}$$

Note that the β_T may grow arbitrarily large as T grows and so this condition in no way enforces stability on the operator. Also note that if H is linear, then this condition follows from the existence and uniqueness conditions.

We now give a convergence proof for discrete time. A proof for continuous time would follow similarly, with the main induction over subsequent intervals $\Delta(I)$ rather than at each index, and is therefore omitted.

Theorem 3. *Suppose that $H : \ell_e \to \ell_e$ satisfies the conditions of Theorem 2, as well as Condition* (1)*, and that we are given $r \in \ell_e$. Then the iteration*

$$y^{(n+1)} \;=\; r - Hy^{(n)}$$

converges to y, the unique solution to

$$y \;=\; r - Hy$$

Proof. The solution y is obviously a fixed point of this iteration, and its truncation $P_k y$ is a fixed point of the following iteration, which simply truncates the input and output of the main iteration.

Define $T_k : P_k\ell_e \to P_k\ell_e$ as

$$\begin{aligned} y &= P_k\left(r - (H(P_k y))\right) \\ &= T_k\left(P_k y\right) \end{aligned}$$

Note that this iteration differs from that of Theorem 2 in that we are iterating over all indexes up to k together, not just one index at a time.

For $k = 0$, however, they are the same, and so we know that $lim_{n\to\infty} P_0 y^{(n)}$ converges to $P_0 y$ on $P_0\ell_e$.

Now for the inductive step, we assume that $lim_{n\to\infty} P_k y^{(n)}$ converges to $P_k y$ on $P_k\ell_e$.

We first prove a contractive property of T_{k+1}. To ease the notation, we will hereafter drop the subscripts and assume that T refers to T_{k+1}, β refers to β_{k+1}, and γ refers to γ_{k+1}. We also assume that signals are truncated with P_{k+1}, so that we only need to specifically mention truncations for earlier indices.

Note that $T^n y^{(0)} = y^{(n)}$, and we will use them interchangeably.

Choose an α such that $0 < \alpha < 1$. Given any $y^{(0)}, \tilde{y}^{(0)} \in \ell_e$ such that $y^{(0)} \neq \tilde{y}^{(0)}$, we may, by the inductive assumption, choose N such that

$$\left\|P_k y^{(n)} - P_k\tilde{y}^{(n)}\right\| \;<\; \frac{\alpha(1-\gamma)}{2\beta}\;\left\|y^{(0)} - \tilde{y}^{(0)}\right\|$$

for all $n \geq N$. Then, for any such n,

$$\begin{aligned} &\left\|Ty^{(n)} - T\tilde{y}^{(n)}\right\| \\ &\quad\leq \left\|H\big(y^{(n)}\big) - H\big(\tilde{y}^{(n)}\big)\right\| \\ &\quad= \left\|H\big(P_k y^{(n)} + y^{(n)}_{k+1}\big) - H\big(P_k\tilde{y}^{(n)} + \tilde{y}^{(n)}_{k+1}\big)\right\| \\ &\quad\leq \left\|H\big(P_k y^{(n)} + y^{(n)}_{k+1}\big) - H\big(P_k\tilde{y}^{(n)} + y^{(n)}_{k+1}\big)\right\| \\ &\qquad\quad + \left\|H\big(P_k\tilde{y}^{(n)} + y^{(n)}_{k+1}\big) - H\big(P_k\tilde{y}^{(n)} + \tilde{y}^{(n)}_{k+1}\big)\right\| \\ &\quad\leq \frac{\alpha}{2}(1-\gamma)\left\|y^{(0)} - \tilde{y}^{(0)}\right\| + \gamma\left\|y^{(n)} - \tilde{y}^{(n)}\right\| \\ &\quad\leq \rho\max\left\{\alpha\left\|y^{(0)} - \tilde{y}^{(0)}\right\|, \left\|y^{(n)} - \tilde{y}^{(n)}\right\|\right\} \end{aligned}$$

where in the second to last step, we used the continuity condition on the left and the contractiveness property from the existence and uniqueness conditions on the right, and in the last step, we let $\rho = \frac{1}{2}(1-\gamma) + \gamma < 1$.

Case (1): If the first quantity that we take the maximum over is larger, then we have shown that

$$\begin{aligned} &\left\|T^{n+1}y^{(0)} - T^{n+1}\tilde{y}^{(0)}\right\| \\ &\qquad= \left\|Ty^{(n)} - T\tilde{y}^{(n)}\right\| < \alpha\left\|y^{(0)} - \tilde{y}^{(0)}\right\| \end{aligned}$$

Case (2): If, on the other hand, the second quantity is larger, then we have shown that

$$\left\|Ty^{(n)} - T\tilde{y}^{(n)}\right\| \leq \rho\left\|y^{(n)} - \tilde{y}^{(n)}\right\| \tag{2}$$

We can further see that, for any $i \in \mathbb{Z}_+$, as long as

$$\alpha\left\|y^{(0)} - \tilde{y}^{(0)}\right\| \;\leq\; \left\|T^i y^{(n)} - T^i\tilde{y}^{(n)}\right\| \tag{3}$$

then we have

$$\left\|T^{i+1}y^{(n)} - T^{i+1}\tilde{y}^{(n)}\right\| \;\leq\; \rho\left\|T^i y^{(n)} - T^i\tilde{y}^{(n)}\right\|$$

Thus if (3) is satisfied for all $0 \leq i \leq j-1$, then

$$\left\|T^j y^{(n)} - T^j\tilde{y}^{(n)}\right\| \;\leq\; \rho^j\left\|y^{(n)} - \tilde{y}^{(n)}\right\| \tag{4}$$

There must exist $m \in \mathbb{Z}_+$ such that

$$\rho^m \;\leq\; \frac{\alpha\left\|y^{(0)} - \tilde{y}^{(0)}\right\|}{\left\|y^{(n)} - \tilde{y}^{(n)}\right\|} \tag{5}$$

Case (2a): If (3) fails for some $0 \leq i \leq m-1$, then we have

$$\begin{aligned} &\left\|T^{n+i}y^{(0)} - T^{n+i}\tilde{y}^{(0)}\right\| \\ &\qquad= \left\|T^i y^{(n)} - T^i\tilde{y}^{(n)}\right\| < \alpha\left\|y^{(0)} - \tilde{y}^{(0)}\right\| \end{aligned}$$

Case (2b): If (3) holds for all $0 \leq i \leq m-1$, then by (4) and (5) we have

$$\begin{aligned} \left\|T^{m+n}y^{(0)} - T^{m+n}\tilde{y}^{(0)}\right\| &\;=\; \left\|T^m y^{(n)} - T^m\tilde{y}^{(n)}\right\| \\ &\;\leq\; \rho^m\left\|y^{(n)} - \tilde{y}^{(n)}\right\| \;\leq\; \alpha\left\|y^{(0)} - \tilde{y}^{(0)}\right\| \end{aligned}$$

We have thus shown that any $y^{(0)}, \tilde{y}^{(0)} \in \ell_e$, there exists $n \in \mathbb{Z}_+$ such that

$$\left\|T^n y^{(0)} - T^n\tilde{y}^{(0)}\right\| \;\leq\; \alpha\left\|y^{(0)} - \tilde{y}^{(0)}\right\| \tag{6}$$

Note that the mapping thus satisfies a condition that is weaker than contractive in the sense of Banach [3], but stronger than weakly contractive in the sense of Bailey [2].

It is clear from this condition that the fixed point must be unique, and we can now prove convergence to this fixed point.

Given any $y^{(0)} \in \ell_e$ and any $\epsilon > 0$, we can apply (6) repeatedly until

$$\left\|y^{(n)} - y\right\| \;=\; \left\|T^n y^{(0)} - T^n y\right\| \;<\; \epsilon \tag{7}$$

So we know that we can always get arbitrarily close to the fixed point y, the remaining question is whether we can stay there.

We choose $\tilde{N}$ large enough so that

$$\left\|P_k y^{(n)} - P_k y\right\| \; < \; \frac{1-\gamma}{2\beta}\,\epsilon \tag{8}$$

for all $n \geq \tilde{N}$, which we may do by the inductive assumption. Applying (6) repeatedly, we find the first $N \geq \tilde{N}$ such that (7) holds.

We then have an N such that (7) holds for $n = N$, and (8) holds for all $n \geq N$.

Now assume that (7) holds for some $n \geq N$. Applying the same steps as before,

$$\begin{aligned} \left\|y^{(n+1)} - y\right\| &= \left\|Ty^{(n)} - Ty\right\| \\ &\leq \rho \max\left\{\epsilon, \left\|y^{(n)} - y\right\|\right\} \\ &< \epsilon \end{aligned}$$

Thus $\left\|y^{(n)} - y\right\| < \epsilon$ for all $n \geq N$, and so $y^{(n)}$ converges to y.

We have now shown that $P_k y^{(n)}$ converges to $P_k y$ for all $k \in \mathbb{Z}_+$, and thus, $y^{(n)}$ converges to y in ℓ_e. ■

3.4 Convergence of operators

We hereafter assume that all operators satisfy the conditions for existence and uniqueness in Theorem 1 or Theorem 2 as well as the continuity condition (1).

Given $K : \mathcal{L}_e \to \mathcal{L}_e$ and $G : \mathcal{L}_e \to \mathcal{L}_e$ define $Y : \mathcal{L}_e \to \mathcal{L}_e$ such that for any $r \in \mathcal{L}_e$

$$y \; = \; Yr$$

is the unique solution to

$$y \; = \; r + GKy$$

and define $R : \mathcal{L}_e \to \mathcal{L}_e$ such that for any $r \in \mathcal{L}_e$

$$u \; = \; Rr \; = \; KYr$$

which is then the unique solution to

$$u \; = \; K(r + Gu)$$

Theorem 4. *Given $K : \mathcal{L}_e \to \mathcal{L}_e$ and $G : \mathcal{L}_e \to \mathcal{L}_e$, $\lim_{n\to\infty} Y^{(n)} = Y = (I - GK)^{-1}$ and $\lim_{n\to\infty} R^{(n)} = R = K(I - GK)^{-1}$.*

Proof. Consider any $r \in \mathcal{L}_e$. The equation

$$y \; = \; r + GKy$$

has a unique solution for y by Theorem 1 and Theorem 2, and $y = Yr$. Thus $(I - GK)$ is bijective, and $Y = (I - GK)^{-1}$ exists.

We then have

$$\lim_{n\to\infty} Y^{(n)} r \; = \; \lim_{n\to\infty} y^{(n)} \; = \; y \; = \; Yr$$

where the inner equality follows from Theorem 3 and the other two follow by definition.

Thus $Y^{(n)} r$ converges to Yr for all $r \in \mathcal{L}_e$, and so $Y^{(n)}$ converges to Y in the strong operator topology.

Then, noting that the continuity of K follows from the technical conditions, we have

$$\begin{aligned} \lim_{n\to\infty} R^{(n)} &= \lim_{n\to\infty} KY^{(n)} \; = \; K \lim_{n\to\infty} Y^{(n)} \\ &= KY \; = \; R \; = \; K(I - GK)^{-1} \end{aligned}$$

■

4 Invariance

The following is the main result of this paper. It introduces a new condition, and shows that constraints satisfying this condition are invariant under feedback.

Theorem 5. *Suppose that S is closed and that*

$$K_1(I \pm GK_2) \in S \qquad \forall\, K_1, K_2 \in S \tag{9}$$

Then

$$\left\{ K(I - GK)^{-1} \mid K \in S \right\} \; = \; S$$

Proof. Suppose that $K \in S$. Then $R^{(0)} = K \in S$. If we assume that $R^{(n)} \in S$, then it follows from Condition (9) that

$$R^{(n+1)} \; = \; K(I + GR^{(n)}) \; \in \; S$$

Thus $R^{(n)} \in S$ for all $n \in \mathbb{Z}_+$. We know from Theorem 4 that $\lim_{n\to\infty} R^{(n)} = R$, and S is closed, so $K(I - GK)^{-1} = R \in S$.

Now, for any $R \in S$, we choose $K = R(I + GR)^{-1}$ such that $K(I - GK)^{-1} = R$. We can then show in the same way as for the above that $K = \lim_{n\to\infty} K^{(n)}$, where $K^{(0)} = R \in S$ and

$$K^{(n+1)} \; = \; R(I - GK^{(n)}) \; \in \; S$$

and thus $K \in S$. ■

5 Parameterization

In this section we consider the parameterization of all stabilizing controllers.

We restate the main result of [1], which gives a parameterization of all stabilizing controllers for a strongly stabilizable plant, with appropriate sign changes for positive feedback.

Theorem 6. *If $K_{\mathrm{nom}} : L_{1e} \to L_{2e}$ is inc. stable, K_{nom} f.g. stabilizes $G : L_{2e} \to L_{1e}$, and*

$$\tilde{G} \; = \; G(I - K_{\mathrm{nom}} G)^{-1}$$

is inc. stable, then

$$\left\{K : L_{1e} \to L_{2e} \mid K \text{ f.g. stabilizes } G\right\} =$$
$$\left\{K_{\text{nom}} + Q(I + \tilde{G}Q)^{-1} \mid Q : L_{1e} \to L_{2e},\ Q \text{ f.g. stable}\right\}$$

Note that when G is inc. stable, we may choose $K_{\text{nom}} = 0$ and this reduces to the following main result of [4].

Theorem 7. *If $G : L_{2e} \to L_{1e}$ is inc. stable, then*

$$\{K : L_{1e} \to L_{2e} \mid K \text{ f.g. stabilizes } G\} = \left\{Q(I + GQ)^{-1} \mid Q : L_{1e} \to L_{2e},\ Q \text{ f.g. stable}\right\}$$

The following theorem then shows how to parameterize all stabilizing controllers for a stable plant subject to a constraint which satisfies our new condition.

Theorem 8. *If $G : \mathcal{L}_e \to \mathcal{L}_e$ is inc. stable, and S is a closed set of admissible controllers which satisfies Condition* (9)*, then*

$$\{K : \mathcal{L}_e \to \mathcal{L}_e \mid K \text{ f.g. stabilizes } G,\ K \in S\} =$$
$$\left\{Q(I+GQ)^{-1} \mid Q : \mathcal{L}_e \to \mathcal{L}_e,\ Q \text{ f.g. stable},\ Q \in S\right\}$$

Proof. Follows from Theorem 7 and Theorem 5. ∎

5.1 Unstable Plant

We would like to use Theorem 6, along with our main result, Theorem 5, to similarly parameterize all constrained stabilizing controllers for an unstable plant.

Assuming that the plant is strongly stabilizable and that the conditions of Theorem 6 are satisfied with a $K_{\text{nom}} \in S$, we seek conditions under which we can make the following statement.

$$\left\{K : \mathcal{L}_e \to \mathcal{L}_e \mid K \text{ f.g. stabilizes G},\ K \in S\right\} = \left\{K_{\text{nom}} + Q(I + \tilde{G}Q)^{-1} \mid Q : \mathcal{L}_e \to \mathcal{L}_e,\ Q \text{ f.g. stable},\ Q \in S\right\}$$

The assumption that S is a subspace is now introduced, since the controller needs to remain in S when K_{nom} is added. We also define, for a constraint set S, a complimentary set of plants $S^\star$ under which our condition is satisfied

$$S^\star = \{G \mid K_1(I \pm GK_2) \in S \ \forall\, K_1, K_2 \in S\}$$

We now list several conditions, any of which allow us to make the parameterization above. They are listed in order such that each condition follows from those below it. Which one is the most useful, which if any are equivalent to each other, and which follow from the main condition (9), or what assumptions are needed for that to be so, are important areas of future work. While slightly more cumbersome to state, number 2 has been the most useful thusfar [6].

1. $K_1(I \pm \tilde{G}K_2) \in S \quad \forall\, K_1, K_2 \in S$ (i.e. $\tilde{G} \in S^\star$)
2. $G_1(I \pm K_{\text{nom}}G_2) \in \check{S} \quad \forall\, G_1, G_2 \in \check{S}$ for any closed $\check{S}$ such that $G \in \check{S} \subseteq S^\star$
3. $(I + K_{\text{nom}}G_1)K_2 \in S \quad \forall\, K_2 \in S,\ G_1 \in S^\star$ and $G \in S^\star$
4. $K_1GK_2 \in S \quad \forall\, K_1, K_2 \in S$ and K linear $\quad \forall\, K \in S$

6 Conclusions

A new condition (9) was introduced relating a plant to constraints imposed on a controller. It was shown in Theorem 5 that when this condition holds, the constraints are invariant under feedback, and the only main assumption was the causality of the operators.

Then in Theorem 8, a parameterization was obtained of all stabilizing constrained controllers subject to the specified constraints, when the plant is stable and our condition is met, and we also presented a hierarchy of conditions which allow us to do the same for an unstable plant.

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