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Parametrization of Stabilizing Controllers Subject to Subspace Constraints

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Abstract— **We consider the problem of designing optimal stabilizing decentralized controllers subject to arbitrary subspace constraints. Recent work considered structural constraints, where each part of the controller has access to some measurements but not others, and developed a parametrization of the stabilizing structured controllers such that the objective is a convex function of the parameter, with the parameter subject to a single quadratic equality constraint. Here we further show that other types of controller constraints arising in decentralized control and other areas of constrained control can be simultaneously encapsulated in this framework with one additional quadratic equality constraint.**

I. Introduction

This paper addresses the problem of optimal decentralized control, where we have multiple controllers, each of which may have access to different information. Most conventional controls analysis breaks down when such decentralization is enforced. Finding optimal controllers when different controllers can access different measurements is notoriously difficult even for the simplest such problem [1], and there are results proving computational intractability for the more general case [2], [3].

When a condition called quadratic invariance holds, which relates these information constraints on the controller to the system being controlled, then the optimal decentralized control problem may be recast as a convex optimization problem [4]. For a particular Youla parametrization, which converts the closed-loop performance objective into a convex function of the new parameter, the information constraint becomes an affine constraint on the parameter, and thus the resulting problem is still convex. The problem of finding the best block diagonal controller however, which represents the case where each subsystem controller may only access measurements from its own subsystem, is never quadratically invariant except for the case where the plant is block diagonal as well; that is, when subsystems do not affect one another.

The parametrization and optimization of stabilizing block diagonal controllers was addressed in a similar fashion using Youla parametrization in [5], focusing on the 2-channel (or 2-block, 2-subsystem, etc.) case. The parametrization similarly converts the objective into a convex function, but the block diagonal constraint on the controller then becomes a quadratic equality constraint on the otherwise free parameter. It is further suggested that the trick for achieving this can be implemented $n-1$ times for n-channel control, resulting in $n-1$ quadratic equality constraints.

While this constraint causes the resulting problem to be nonconvex, it still converts a generally intractable problem into one where the only difficulty is a well-understood type of constraint. Solving this resulting constrained problem is then further explored in [6], and many other methods exist for addressing quadratic equality constraints.

Recent work [7] addressed arbitrary structural constraints, which are not generally block diagonal nor quadratically invariant. It was first discussed how to convert this general problem to a block diagonal synthesis problem. It was then shown that the key insight of [5] could be adapted to similarly convert the block diagonal synthesis problem into a problem on a stable Youla parameter with a convex objective, but subject to a single quadratic equality constraint, regardless of the number of blocks.

This paper considers additional subspace constraints on the controller, as can arise in decentralized control design, and handles them within the same framework. We see that delay constraints, such as those which arise when each part of the controller must wait different amounts of time before accessing different measurements, are seamlessly incorporated once the problem has been diagonalized, and do not affect the form of the resulting optimization problem. We see that constraints which require certain parts of the controller to behave equivalently to other parts of the controller, such as those which arise in simultaneous control or symmetric control, result in a second quadratic equality constraint.

II. Preliminaries

We suppose that we have a generalized plant $P \in \mathcal{R}_p$ partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & G \end{bmatrix}$$

We define the ***closed-loop map*** by

$$f(P,K) = P_{11} + P_{12}K(I - GK)^{-1}P_{21}$$

The map $f(P,K)$ is also called the (lower) ***linear fractional transformation*** (LFT) of P and K. Note that we abbreviate $G = P_{22}$, since we will refer to that block frequently, and so that we may refer to its subdivisions without ambiguity. This interconnection is shown (along with disturbances) in Figure 1.

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We suppose that there are n_y sensor measurements and n_u control actions, and thus partition the sensor measurements and control actions as

$$y = \begin{bmatrix} y_1^T & \dots & y_{n_y}^T \end{bmatrix}^T \qquad u = \begin{bmatrix} u_1^T & \dots & u_{n_u}^T \end{bmatrix}^T$$

with partition sizes

$$y_i \in L_2^{p_i} \ \forall\, i \in 1,\dots,n_y \qquad u_j \in L_2^{m_j} \ \forall\, j \in 1,\dots,n_u$$

and total sizes

$$y \in L_2^p, \quad \sum_{i=1}^{n_y} p_i = p; \qquad u \in L_2^m, \quad \sum_{j=1}^{n_u} m_j = m$$

and then further partition G and K as

$$G = \begin{bmatrix} G_{11} & \dots & G_{1n_u} \\ \vdots & & \vdots \\ G_{n_y1} & \dots & G_{n_yn_u} \end{bmatrix} \qquad K = \begin{bmatrix} K_{11} & \dots & K_{1n_y} \\ \vdots & & \vdots \\ K_{n_u1} & \dots & K_{n_un_y} \end{bmatrix}$$

with block sizes

$$G_{ij} \in \mathcal{R}_{sp}^{p_i \times m_j}, \qquad K_{ji} \in \mathcal{R}_p^{m_j \times p_i} \ \ \forall\, i,j$$

and total sizes

$$G \in \mathcal{R}_{sp}^{p\times m} \qquad K \in \mathcal{R}_p^{m\times p}$$

This will typically represent n subsystems, each with its own controller, in which case we will have $n = n_y = n_u$, but this does not have to be the case.

We denote by $\mathcal{R}_p^{m\times n}$ the set of matrix-valued real-rational proper transfer matrices, by $\mathcal{R}_{sp}^{m\times n}$ the set of matrix-valued real-rational strictly proper transfer matrices, and by $\mathcal{RH}_\infty^{m\times n}$ the set of real-rational proper stable transfer matrices, omitting the superscripts when the dimensions are implied by context.

Let I_n represent the $n \times n$ identity.

A. Stabilization

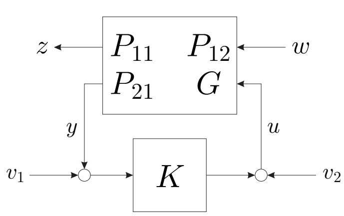

Fig. 1. Linear fractional interconnection of P and K

We say that K ***stabilizes*** P if in Figure 1 the nine transfer matrices from w, v_1, v_2 to z, u, y belong to $\mathcal{RH}_\infty$. We say that K ***stabilizes*** G if in the figure the four transfer matrices from v_1, v_2 to u, y belong to $\mathcal{RH}_\infty$. P is called ***stabilizable*** if there exists $K \in \mathcal{R}_p^{m\times p}$ such that K stabilizes P. The following standard result relates stabilization of P with stabilization of G.

Theorem 1: Suppose $G \in \mathcal{R}_{sp}^{p\times m}$ and $P \in \mathcal{R}_p^{(n_z+p)\times(n_w+m)}$, and suppose P is stabilizable. Then K stabilizes P if and only if K stabilizes G.

Proof: See, for example, Chapter 4 of [8]. ■

For a given system P, all controllers that stabilize the system may be parameterized using the well-known Youla parametrization [9], stated below.

Theorem 2: Suppose that we have a doubly coprime factorization of G over $\mathcal{RH}_\infty$, that is, $M_l, N_l, X_l, Y_l, M_r, N_r, X_r, Y_r \in \mathcal{RH}_\infty$ such that $G = N_r M_r^{-1} = M_l^{-1} N_l$ and

$$\begin{bmatrix} X_l & -Y_l \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_r \\ N_r & X_r \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}. \tag{1}$$

Then the set of all stabilizing controllers is given by

$$\begin{aligned} &\{K \in \mathcal{R}_p \mid K \text{ stabilizes } G\} \\ &\quad= \Big\{(Y_r - M_rQ)(X_r - N_rQ)^{-1} \mid \\ &\qquad\quad X_r - N_rQ \text{ is invertible}, Q \in \mathcal{RH}_\infty\Big\} \\ &\quad= \Big\{(X_l - QN_l)^{-1}(Y_l - QM_l) \mid \\ &\qquad\quad X_l - QN_l \text{ is invertible}, Q \in \mathcal{RH}_\infty\Big\}. \end{aligned}$$

Furthermore, the set of all closed-loop maps achievable with stabilizing controllers is

$$\begin{aligned} &\Big\{ f(P,K) \mid K \in \mathcal{R}_p,\ K \text{ stabilizes } P \Big\} \\ &= \Big\{ T_1 - T_2QT_3 \mid X_r - N_rQ \text{ is invertible},\ Q \in \mathcal{RH}_\infty \Big\}, \end{aligned} \tag{2}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$ are given by

$$\begin{aligned} T_1 &= P_{11} + P_{12}Y_rM_lP_{21} \\ T_2 &= P_{12}M_r \\ T_3 &= M_lP_{21}. \end{aligned} \tag{3}$$

Proof: See, for example, Chapter 4 of [8]. ■

The parameter Q is usually referred to as the Youla parameter. The following lemma shows that the two parametrizations above give the same change of variables. The proof may be found in [7], in which it follows similarly from [5].

Lemma 3: Suppose $G \in \mathcal{R}_{sp}$. Then the set of all stabilizing controllers is given by

$$\begin{aligned} &\{K \in \mathcal{R}_p \mid K \text{ stabilizes } P\} = \\ &\Big\{(Y_r - M_rQ)(X_r - N_rQ)^{-1} = (X_l - QN_l)^{-1}(Y_l - QM_l) \\ &\qquad\qquad \mid Q \in \mathcal{RH}_\infty\Big\}. \end{aligned}$$

Remark 4: Even if G is not strictly proper, the invertibility conditions still hold for almost all parameters Q [10, p.111].

III. Problem Formulation

We develop the optimization problem that we need to solve for optimal synthesis subject to three types of subspace constraint.

A. Structural Constraints

Structural constraints, which specify that each controller may access certain sensor measurements but not others, manifest themselves as sparsity constraints on the controller to be designed. We here introduce some notation for representing this type of constraint.

Let $\mathbb{B} = \{0, 1\}$ represent the set of binary numbers. Suppose $A^{\text{bin}} \in \mathbb{B}^{m\times n}$ is a binary matrix. We define the subspace

$$\text{Sparse}(A^{\text{bin}}) = \Big\{ B \in \mathcal{R}_p \mid B_{ij}(j\omega) = 0 \text{ for all } i, j \text{ such that } A^{\text{bin}}_{ij} = 0 \text{ for almost all } \omega \in \mathbb{R} \Big\}$$

giving all of the proper transfer function matrices which satisfy the given sparsity constraint.

We then represent the constraints on the overall controller with a binary matrix $K^{\text{bin}} \in \mathbb{B}^{n_u\times n_y}$ where

$$K^{\text{bin}}_{kl} = \begin{cases} 1, & \text{if control input } k \\ & \text{may access sensor measurement } l \\ 0, & \text{if not.} \end{cases}$$

The subspace of controllers satisfying the structural constraint is then given as

$$S_t = \text{Sparse}(K^{\text{bin}}).$$

B. Composition Constraints

For an active part of the controller, that is, for k, l such that $K^{\text{bin}}_{kl} = 1$, a composition constraint imposes that it must take the form $K_{kl} = \tilde{V}_{kl}\hat{K}_{kl}\tilde{U}_{kl}$ for some given operators $\tilde{V}_{kl}$ and $\tilde{U}_{kl}$, where $\hat{K}_{kl}$ is then left to be designed. The motivating example for this type of constraint is the case where there is a transmission delay before the kth controller can access the lth sensor measurement y_l, and then $\tilde{U}_{kl}$ would be the necessary delay, but we can handle it much more generally.

C. Scaling Constraints

We lastly consider constraints which impose that one part of the controller must be a multiple of another. The motivating examples come from the special case requiring various parts of the controller to be equivalent. These include simultaneous control, where the controller must be block diagonal, and then each block must be equivalent, as well as symmetric control, where each part of the controller must be equivalent to its transposed counterpart, except for the diagonal blocks, which have no such constraint.

These constraints generally take the form

$$K^{\text{lmult}}_{kl} K_{kl} = K_{ij} K^{\text{rmult}}_{ij} \tag{4}$$

for some $K^{\text{lmult}} \in \mathbb{R}^{m_i\times m_k}$ and $K^{\text{rmult}} \in \mathbb{R}^{p_j\times p_l}$. For most applications of interest, both multipliers will be the identity, and note that this is only possible when we have $m_i = m_k$ and $p_j = p_l$. When no such constraint needs to be enforced for given i, j, k, l, we assign $K^{\text{lmult}}_{kl} = 0$ and $K^{\text{rmult}}_{ij} = 0$.

We define S_c as the set of all controllers satisfying these scaling constraints.

We can now set up our main problem of finding the best controller subject to all of these subspace constraints.

D. Problem Setup

Given a generalized plant P and a subspace of admissible controllers S, we would then like to solve the following problem:

$$\begin{aligned} \text{minimize} \quad & \|f(P, K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned} \tag{5}$$

Here $\|\cdot\|$ is any norm on the closed-loop map chosen to encapsulate the control performance objectives. The subspace of admissible controllers, S, is defined to encapsulate the constraints outlined above on which controllers can access which sensor measurements, on the pre- and post-processing of signals into and out of each part of the controller, and on which parts of the controller must be multiples of others.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved with many standard techniques. Note that the cost function $\|f(P, K)\|$ is in general a non-convex function of K. If the information constraint is quadratically invariant [4] with respect to the plant, then the problem may be recast as a convex optimization problem, but no computationally tractable approach is known for solving this problem for arbitrary P and S.

IV. Diagonalization

In this section, we discuss how the problem of finding the optimal structured controller $K \in S_t$ can be converted to a problem of finding an optimal block diagonal controller.

The concepts in this section are fairly straightforward, but covering the general case rigorously is unfortunately of a tedious nature requiring multiple subscripts. We will summarize the transformation here and provide an example, and details can be found in Section IV of [7].

Given the structural constraints, we let a be the total number of active blocks (for which $K^{\text{bin}}_{kl} = 1$), and then for each of these active blocks, we assign a unique $\alpha \in \{1, \ldots, a\}$, and let $\kappa_\alpha = k$ capture the control input associated with that active block and let $\lambda_\alpha = l$ capture the sensor measurement associated with that active block.

We are then able to define a left-invertible matrix of 1's and 0's $U \in \mathbb{R}^{a_y\times p}$ that repeats sensor measurements as necessary such that $\tilde{y} = Uy$ gives the measurements for the block-diagonal controller, and to define a right-invertible matrix of 1's and 0's $V \in \mathbb{R}^{m\times a_u}$ that reconstitutes the output from the diagonal controller as the controller inputs to the plant as $u = V\tilde{u}$.

We can then define a new generalized plant $\tilde{P} \in \mathcal{R}_p^{(n_z+a_y)\times(n_w+a_u)}$ with the following components

$$\begin{aligned} \tilde{P}_{11} &= P_{11} & \tilde{P}_{12} &= P_{12}V \\ \tilde{P}_{21} &= UP_{21} & \tilde{G} &= UGV \end{aligned} \tag{6}$$

which maps $(w, \tilde{u}) \to (z, \tilde{y})$.

Example 5: Suppose we are trying to find the best controller $K \in S_t$ where $S_t = \text{Sparse}(K^{\text{bin}})$ and where the admissible controller structure is given by

$$K^{\text{bin}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

that is, we need to find the best 3×4 controller where only 5 particular parts of the controller may be active.

We then have $a = 5$, and assign $(\kappa_1, \lambda_1) = (1, 1)$, $(\kappa_2, \lambda_2) = (3, 1)$, $(\kappa_3, \lambda_3) = (2, 2)$, $(\kappa_4, \lambda_4) = (1, 3)$, and $(\kappa_5, \lambda_5) = (3, 4)$. We then get

$$U = \begin{bmatrix} I & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \qquad V = \begin{bmatrix} I & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & I \end{bmatrix}$$

such that $\tilde{y} = Uy = \begin{bmatrix} y_1^T & y_1^T & y_2^T & y_3^T & y_4^T \end{bmatrix}^T$ and such that $u = V\tilde{u} = \begin{bmatrix} (\tilde{u}_1 + \tilde{u}_4)^T & \tilde{u}_3^T & (\tilde{u}_2 + \tilde{u}_5)^T \end{bmatrix}^T$. The matrix U repeats the first sensor measurement since the first two active parts of the controller, and thus the first two parts of the block diagonal controller, both need to access it, and then the matrix V takes the 5 signals from the block diagonal controller ($\tilde{u}$) and reconstitutes the 3 controller inputs (u).

We may then replace the given generalized plant P with $\tilde{P}$ as in (6), and any admissible controller $K \in S_t$ by the block diagonal controller $\tilde{K} = \text{diag}(K_{\kappa_1\lambda_1}, \ldots, K_{\kappa_5\lambda_5}) = \text{diag}(K_{11}, K_{31}, K_{22}, K_{13}, K_{34})$ such that $K = V\tilde{K}U$, and this maintains the same closed-loop map.

Having defined these transformations, we then need to show that they indeed yield a diagonal synthesis problem which is equivalent to our original problem.

We first give a lemma which verifies that we have properly set up our bijection from admissible structured controllers $K \in S_t$ to block diagonal controllers $\tilde{K} \in \tilde{S}_d$. The proof may be found in [7].

Lemma 6: Given $\tilde{K} \in \tilde{S}_d$, we can let $K = V\tilde{K}U$, and then $K \in S_t$.

Given $K \in S_t$, we can let

$$\tilde{K}_{\alpha\beta} = \begin{cases} K_{kl}, & \text{if } \alpha = \beta, k = \kappa_\alpha, l = \lambda_\alpha \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

and then $\tilde{K} \in \tilde{S}_d$ and $K = V\tilde{K}U$.

Remark 7: Note that given $K \in S_t$, $V^\dagger K U^\dagger$ is generally not in $\tilde{S}_d$.

We now define the subspace of controllers in the new space which satisfy the scaling constraints in the original space as $\tilde{K} \in \tilde{S}_c$ iff

$$K^{\text{lmult}}_{\kappa_\alpha\lambda_\alpha} \tilde{K}_{\alpha\alpha} = \tilde{K}_{\beta\beta} K^{\text{rmult}}_{\kappa_\beta\lambda_\beta} \quad \forall\, \alpha, \beta \in 1, \ldots, a \tag{8}$$

and show that, given a controller satisfying the structural constraints, this is indeed equivalent to the original scaling constraints.

Lemma 8: Given the same transformation between K and $\tilde{K}$ stated in Lemma 6,

$$K \in S_t \cap S_c \qquad \Leftrightarrow \qquad \tilde{K} \in \tilde{S}_d \cap \tilde{S}_c.$$

Proof: We have $K \in S_t \Leftrightarrow \tilde{K} \in \tilde{S}_d$ from Lemma 6, and then the equivalence of $K \in S_c$ and $\tilde{K} \in \tilde{S}_c$ given that $\tilde{K} \in \tilde{S}_d$ is follows directly from (4),(7), (8). ■

If $K = V\tilde{K}U$, then

$$\begin{aligned} f(P, K) &= P_{11} + P_{12}(V\tilde{K}U)(I - GV\tilde{K}U)^{-1}P_{21} \\ &= P_{11} + (P_{12}V)\tilde{K}(I - UGV\tilde{K})^{-1}UP_{21} \\ &= \tilde{P}_{11} + \tilde{P}_{12}\tilde{K}(I - \tilde{G}\tilde{K})^{-1}\tilde{P}_{21} \\ &= f(\tilde{P}, \tilde{K}) \end{aligned}$$

where we used the push-through identity in the second step, and thus the closed-loop maps are identical.

With repeated use of the push-through identity we also find

$$\begin{bmatrix} I & K \\ G & I \end{bmatrix}^{-1} = \begin{bmatrix} V & 0 \\ 0 & U^\dagger \end{bmatrix} \begin{bmatrix} I & \tilde{K} \\ \tilde{G} & I \end{bmatrix}^{-1} \begin{bmatrix} V^\dagger & 0 \\ 0 & U \end{bmatrix}$$

Thus if $\tilde{K}$ stabilizes $\tilde{G}$, then K stabilizes G. The converse does not generally follow from this relation, but if an unstable mode is suppressed by $V, U^\dagger, V^\dagger$, or U, we just need a stabilizing $\tilde{K}$ which yields $V\tilde{K}(I - \tilde{G}\tilde{K})^{-1}U = K(I - GK)^{-1}$ to achieve the same closed-loop map. Generalizing the technical conditions needed for this, or understanding in what manner enforcing internal stability on the diagonalized problem represents a stronger notion of stability enforced on the orignal problem, is ongoing work.

We lastly define $\tilde{S} = \tilde{S}_d \cap \tilde{S}_c$ as the set of admissible controllers after the transformation developed in this section; that is, the set of controllers which are block diagonal, corresponding to the structural constraint, and which also satisfy the scaling constraints.

V. Parametrization

In this section, we assume that our problem has been converted to one of finding an optimal block diagonal controller, and address the problem of parametrizing all of the stabilizing controllers and all of the achievable closed-loop maps, to further transform the optimization problem to one over a stable parameter with a convex objective. We show that this can be achieved, with the diagonalization manifesting itself as a quadratic equality constraint on the parameter, and the scaling constraints manifesting themselves as a second quadratic equality constraint on the paramter.

A. Composition Constraints

Since the problem has been diagonalized, the constraints of the type described in Section III-B can be easily handled by moving the given operators over to the plant. The matrix U (and thus $\tilde{P}_{21}$ and $\tilde{G}$) are left-multiplied by $\text{diag}(\tilde{U}_{\kappa_1\lambda_1}, \ldots, \tilde{U}_{\kappa_a\lambda_a})$, and the matrix V (and thus $\tilde{P}_{12}$ and $\tilde{G}$) are right-multiplied by $\text{diag}(\tilde{V}_{\kappa_1\lambda_1}, \ldots, \tilde{V}_{\kappa_a\lambda_a})$. This preserves the closed-loop map for any designed controller, though it should be noted that stabilizability can be affected if more general operators than the motivating ones are used in the compositions. This transformation leaves us with a controller to design which must be block diagonal and which must satisfy the scaling constraints, so we now turn our attention to those constraints, and we do not discuss the composition constraints any further, as they are embedded in the plant.

B. Equivalent Diagonal Constraint

The key insight in this section, that a block diagonal constraint can be expressed as in (9), which then becomes a quadratic constraint in the Youla parameter, is largely derived from Manousiouthakis [5]. There this idea was introduced for 2-channel control (block diagonal with 2 blocks), and it was suggested that the same technique could be used $n-1$ times, resulting in $n-1$ quadratic equality constraints on the Youla parameter, to enforce a block diagonal constraint with n blocks. Here, in addition to having first started with an arbitrary structural constraint, we now show how this parametrization can be achieved with just one quadratic equality constraint, regardless of the number of blocks, as was developed in [7].

Define $L_l \in \mathbb{R}^{a_u \times a_u}$ as

$$L_l = \mathrm{diag}(I_{b_1}, 2I_{b_2}, \ldots, aI_{b_a})$$

and define $L_r \in \mathbb{R}^{a_y \times a_y}$ as

$$L_r = \mathrm{diag}(I_{c_1}, 2I_{c_2}, \ldots, aI_{c_a})$$

and note that the two matrices are identical if all controller blocks are square or scalar.

We now show how these matrices can be used to enforce a block diagonal constraint.

Lemma 9: Given $\tilde{K} \in \mathcal{R}_p^{a_u \times a_y}$,

$$L_l\tilde{K} = \tilde{K}L_r \qquad \Leftrightarrow \qquad \tilde{K} \in \tilde{S}_d. \tag{9}$$

Proof:

$$\begin{aligned} & L_l\tilde{K} = \tilde{K}L_r \\ \Leftrightarrow \quad & \sum_{k=1}^{a}(L_l)_{ik}\tilde{K}_{kj} = \sum_{k=1}^{a}\tilde{K}_{ik}(L_r)_{kj} \quad \forall\, i,j \\ \Leftrightarrow \quad & (L_l)_{ii}\tilde{K}_{ij} = \tilde{K}_{ij}(L_r)_{jj} \quad \forall\, i,j \text{ since } L_l, L_r \text{ diag.} \\ \Leftrightarrow \quad & i\tilde{K}_{ij} = j\tilde{K}_{ij} \quad \forall\, i,j \\ \Leftrightarrow \quad & \tilde{K}_{ij} = 0 \quad \forall\, i \neq j \\ \Leftrightarrow \quad & \tilde{K} \in \tilde{S}_d. \end{aligned}$$

■

C. Equivalent Scaling Constraint

We now define matrices that will allow us to similarly recharacterize the scaling constraints.

Define $A_l \in \mathbb{R}^{a_u \times a_u}$ as

$$(A_l)_{ij} = \begin{cases} I_{b_i}, & \text{if } i=j \\ K^{\text{lmult}}_{\kappa_j\lambda_j}, & \text{otherwise.} \end{cases}$$

and define $A_r \in \mathbb{R}^{a_y \times a_y}$ as

$$(A_r)_{ij} = \begin{cases} I_{c_i}, & \text{if } i=j \\ K^{\text{rmult}}_{\kappa_i\lambda_i}, & \text{otherwise.} \end{cases}$$

We now show how these matrices can be used to enforce a scaling constraint.

Lemma 10: Given $\tilde{K} \in \mathcal{R}_p^{a_u \times a_y}$,

$$L_l\tilde{K} = \tilde{K}L_r, \quad A_l\tilde{K} = \tilde{K}A_r \qquad \Leftrightarrow \qquad \tilde{K} \in \tilde{S} \tag{10}$$

Proof: We know from Lemma 9 that the first equality gives us $\tilde{K} \in \tilde{S}_d$. Then,

$$\begin{aligned} & A_l\tilde{K} = \tilde{K}A_r \\ \Leftrightarrow \quad & \sum_{k=1}^{a}(A_l)_{ik}\tilde{K}_{kj} = \sum_{k=1}^{a}\tilde{K}_{ik}(A_r)_{kj} \quad \forall\, i,j \\ \Leftrightarrow \quad & (A_l)_{ij}\tilde{K}_{jj} = \tilde{K}_{ii}(A_r)_{ij} \quad \forall\, i,j \text{ since } \tilde{K} \text{ diag.} \\ \Leftrightarrow \quad & K^{\text{lmult}}_{\kappa_j\lambda_j}\tilde{K}_{jj} = \tilde{K}_{ii}K^{\text{rmult}}_{\kappa_i\lambda_i} \quad \forall\, i,j \\ \Leftrightarrow \quad & \tilde{K} \in \tilde{S}_c. \end{aligned}$$

■

Example 11: Suppose that we wish to achieve simultaneous control of 4 (square) subsystems. We then have

$$L_l = L_r = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 2I & 0 & 0 \\ 0 & 0 & 3I & 0 \\ 0 & 0 & 0 & 4I \end{bmatrix}$$

so that $L_l\tilde{K} = \tilde{K}L_r$ forces $\tilde{K}$ to be block diagonal, and then

$$A_l = A_r = \begin{bmatrix} I & 0 & 0 & 0 \\ I & I & 0 & 0 \\ 0 & I & I & 0 \\ 0 & 0 & I & I \end{bmatrix}$$

so that $A_l\tilde{K} = \tilde{K}A_r$ forces $\tilde{K}_{11} = \tilde{K}_{22} = \tilde{K}_{33} = \tilde{K}_{44}$.

D. Main Result

The following theorem is the main result of this paper. It shows that all of the stabilizing block diagonal controllers subject to the scaling constraints can be parametrized by a stable Youla parameter, subject to two quadratic equality constraints. It further shows that the set of all achievable closed-loop maps may then be expressed an an affine function of this Youla parameter, subject to the same quadratic equality constraints.

Theorem 12: Suppose that $\tilde{P}$ is stabilizable, and that we have a doubly coprime factorization of $\tilde{G}$ as in (1). Then

$$\begin{aligned} \{\tilde{K} \in \mathcal{R}_p \mid \tilde{K} \text{ stabilizes } \tilde{P},\ \tilde{K} \in \tilde{S}\} \ = \\ \{(Y_r - M_rQ)(X_r - N_rQ)^{-1} \mid Q \in \mathcal{RH}_\infty, \\ q_d(Q) = 0,\ q_s(Q) = 0\} \end{aligned}$$

where

$$q_d(Q) \ = \ \begin{bmatrix} I_{a_u} & Q \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \begin{bmatrix} I_{a_y} \\ Q \end{bmatrix} \tag{11}$$

and

$$q_c(Q) \ = \ \begin{bmatrix} I_{a_u} & Q \end{bmatrix} \begin{bmatrix} W_5 & W_6 \\ W_7 & W_8 \end{bmatrix} \begin{bmatrix} I_{a_y} \\ Q \end{bmatrix} \tag{12}$$

and where

$$\begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} = \begin{bmatrix} X_l & Y_l \\ -N_l & -M_l \end{bmatrix} \begin{bmatrix} L_l & 0 \\ 0 & L_r \end{bmatrix} \begin{bmatrix} Y_r & -M_r \\ -X_r & N_r \end{bmatrix}. \tag{13}$$

and

$$\begin{bmatrix} W_5 & W_6 \\ W_7 & W_8 \end{bmatrix} = \begin{bmatrix} X_l & Y_l \\ -N_l & -M_l \end{bmatrix} \begin{bmatrix} A_l & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} Y_r & -M_r \\ -X_r & N_r \end{bmatrix}. \tag{14}$$

Further,

$$\{f(\tilde{P},\tilde{K}) \mid \tilde{K} \text{ stabilizes } \tilde{P},\ \tilde{K} \in \tilde{S}\} = \{T_1 - T_2 Q T_3 \mid Q \in \mathcal{RH}_\infty,\ q_d(Q) = 0,\ q_s(Q) = 0\} \tag{15}$$

where T_i are given as in (3).

Proof: With this change of variables, the equivalence of $Q \in \mathcal{RH}_\infty$ with $\tilde{K}$ stabilizing $\tilde{P}$, as well as $f(\tilde{P},\tilde{K}) = T_1 - T_2 Q T_3$, follow from Theorem 2. It just remains to show that the diagonal and scaling constraints on the controller are equivalent to the quadratic constraints on the Youla parameter. Note that we utilize Lemma 10 in the first step, and that we utilize Lemma 3 and the equivalence of the left and right parametrizations in the second step:

$$\begin{aligned}
& \tilde{K} \in \tilde{S}, \quad \tilde{K} \in S_c \\
\Leftrightarrow\quad & L_l \tilde{K} = \tilde{K} L_r, \quad A_l \tilde{K} = \tilde{K} A_r \\
\Leftrightarrow\quad & L_l (Y_r - M_r Q)(X_r - N_r Q)^{-1} \\
& \qquad = (X_l - Q N_l)^{-1} (Y_l - Q M_l) L_r, \\
& A_l (Y_r - M_r Q)(X_r - N_r Q)^{-1} \\
& \qquad = (X_l - Q N_l)^{-1} (Y_l - Q M_l) A_r \\
\Leftrightarrow\quad & (X_l - Q N_l) L_l (Y_r - M_r Q) \\
& \qquad = (Y_l - Q M_l) L_r (X_r - N_r Q), \\
& (X_l - Q N_l) A_l (Y_r - M_r Q) \\
& \qquad = (Y_l - Q M_l) A_r (X_r - N_r Q) \\
\Leftrightarrow\quad & \underbrace{(X_l L_l Y_r - Y_l L_r X_r)}_{W_1} + \underbrace{(Y_l L_r N_r - X_l L_l M_r)}_{W_2} Q \\
& + Q \underbrace{(M_l L_r X_r - N_l L_l Y_r)}_{W_3} + Q \underbrace{(N_l L_l M_r - M_l L_r N_r)}_{W_4} Q = 0, \\
& \underbrace{(X_l A_l Y_r - Y_l A_r X_r)}_{W_5} + \underbrace{(Y_l A_r N_r - X_l A_l M_r)}_{W_6} Q \\
& + Q \underbrace{(M_l A_r X_r - N_l A_l Y_r)}_{W_7} + Q \underbrace{(N_l A_l M_r - M_l A_r N_r)}_{W_8} Q = 0 \\
\Leftrightarrow\quad & q_d(Q) = 0, \quad q_s(Q) = 0
\end{aligned}$$

■

We may thus solve the following equivalent problem

$$\begin{aligned}
\text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\
\text{subject to} \quad & Q \in \mathcal{RH}_\infty \\
& q_d(Q) = 0 \\
& q_s(Q) = 0
\end{aligned} \tag{16}$$

to find the optimal Q^*, recover the optimal diagonal controller as $\tilde{K}^* = (Y_r - M_r Q^*)(X_r - N_r Q^*)^{-1}$, and then recover the structured controller for our original problem (5) as $K^* = V \tilde{K}^* U$.

Remark 13: Note that the calculation of the Youla parameters (1) and the closed-loop parameters (3) in Theorem 12 must be based on $\tilde{G}$ and $\tilde{P}$.

VI. CONCLUSIONS

We have considered the problem of synthesizing optimal stabilizing controllers subject to decentralization constraints. We first showed how to recast this as a block diagonal synthesis problem, and then how to recast that as a problem over a stable Youla parameter with a convex objective.

The general problem we are addressing is known to be intractable, and so it is not surprising that the resulting optimization problem is not convex in general. However, we have taken a broad class of important intractable problems, shown how to handle them in a unified manner, and shown how the inherent difficulty of the problem can be concentrated into two quadratic equality constraints.

The synthesis of optimal (decentralized) control via Youla parametrization can now be summarized as follows. Without decentralization constraints, finding the optimal stabilizing controller can be cast as optimizing a convex function of the Youla parameter, where the parameter is free and stable. If the controller is instead subject to a quadratically invariant constraint, the parameter is subject to an affine equality constraint. If the controller is subject to a structural constraint which is not quadratically invariant, the parameter is subject to a quadratic equality constraint. If the controller is subject to additional subspace constraints, the parameter is subject to a second quadratic equality constraint.

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