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An Optimization-based Approach to Decentralized Assignability

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Abstract— **This paper discusses the controllability of linear time-invariant (LTI) systems with decentralized controllers. Whether an LTI system is controllable (by LTI controllers) with respect to a given information structure can be determined by testing for fixed modes, but this gives a binary answer with no information about robustness. Measures have been developed to further determine how far a system is from having a fixed mode, in particular the decentralized assignability measure of Vaz and Davison in 1988, but these measures cannot actually be computed in most cases. We thus seek an easily computable, non-binary measure of controllability for LTI systems with decentralized controllers of arbitrary information structure.**

In this paper, we address this problem by utilizing modern optimization techniques to tackle the decentralized assignability measure. The main difficulties which have previously precluded its widespread use, are that it involves the minimization of the n-th singular value of a matrix, which must further be minimized over a power set of the subsystems. We will propose three methods to address its computation. First, we will discuss a relaxed convex problem, using the nuclear norm in place of the singular value, and expressing the power set minimization as binary constraints which can be relaxed to the hypercube. Our second algorithm simply entails rounding when the first method fails to reach a corner of the hypercube. Our final algorithm is developed using the Alternating Direction Method of Multipliers (ADMM), and is shown to decouple the effects of the binary variables, such that they can be optimized directly with per-iteration computations scaling linearly, rather than exponentially, with the number of subsystems. This final method is shown to produce results which closely track the assignability measure across a variety of fixed mode types.

I. Introduction

A seminal result in decentralized control is the development of fixed modes by [1] - that plant modes which cannot be moved with a static decentralized controller cannot be moved by a dynamic one either, and that the other modes which can be moved can be shifted to any chosen locations with arbitrary precision.

In many cases one needs to know more than just whether or not a fixed mode is present. It could be the case that although the plant is theoretically controllable (i.e., there exist no fixed modes), that a large control effort is required to move the states, and/or that a small perturbation to the plant would result in a fixed mode. These questions have been well answered for the centralized case through controllability, observability, and Hankel operators. In particular, Hankel singular values of a stable plant provide a non-binary measure of how controllable and observable that plant is, and are easy to compute.

In the decentralized case, Vaz & Davison have defined the decentralized assignability measure based on the distance of the plant from the set of plants that have a fixed mode [2]. They characterized and connected the mobility of an eigenvalue of the plant, which is the change in its location when a decentralized controller of bounded magnitude is applied, to the aforementioned measure. They have also proven that this measure would be non-zero if and only if there exist no fixed modes. However, this metric is hard to compute for all but the smallest problems. As an alternative approach, [3] have explored the use of the Hankel operator to develop an easily computable metric which could provide information regarding proximity to a fixed mode for decentralized control, which would work on some but not all the considered cases.

The developed metric by Vaz & Davison in [2] corresponds to the minimization of the n-th singular value over a power set, and hence, would be intractable in general. In this paper, we seek to address the aforementioned measure by providing tractable approximations.

To this end, we will first replace the optimization problem in Vaz & Davison which involves minimizing over a matrix with variable dimension, by a form that would only contain a fixed-size matrix. We then propose three methods for computation of the relaxed version. Currently, in all of these three methods, we use the nuclear norm as a proxy for rank minimization, rather than directly minimizing over the n-th singular value. Our first method would result in a convex program that involves a relaxed real vector instead of the ideal binary one. Secondly, we would take the solution of the first method, and round the relaxed real vector to its closest binary value, and derive a subsequent convex optimization problem. Lastly, we will use the Alternating Direction Method of Multipliers (ADMM) [4] to propose an iterative algorithm that would directly achieve the desired binary value in all the iterations.

The organization of the paper is as follows. We state some preliminary notations and definitions in Section II, and review the fixed modes, the decentralized assignability measure of [2], and the diagonalization method in Section III. We will then form the relaxed version of the optimization problem of the interest, and state its characteristics in Section IV. We will propose three methods pertaining to the relaxed problem in Section V. We will use numerical examples to illustrate, investigate, and compare the performance of the proposed methods in Section VI.

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II. Preliminaries

We assume that we have a causal, linear time-invariant, and strictly proper plant G, with n_u inputs and n_y outputs. A state-space representation for G is given by $(A, B, C, 0)$. We next introduce some notation that will help encapsulate the main type of decentralization we consider below.

We are not only interested in the input-output characteristics of the plant G, but also in imposing structure on the feedback controller. We will denote the feedback controller by K, such that we have $u(s) = K(s)y(s)$, and suppose that the controller is finite dimensional, causal, linear time-invariant, and proper. We are interested in confining the space of admissible controllers by imposing an information constraint on K. We will primarily focus on sparsity constraints, and denote the set of controllers that satisfy the sparsity constraint by $\mathcal{S}$.

We can associate the set of controllers that satisfy some sparsity constraint (i.e., are in $\mathcal{S}$) to a binary matrix $K^{\text{bin}} \in \mathbb{B}^{n_u \times n_y}$, with $\mathbb{B} = \{0, 1\}$, whose (i, j) entry is equal to 1 ($K^{\text{bin}}_{ij} = 1$), if the i-th control input may access the j-th measurement output, and 0 otherwise. We define $a \triangleq \sum_{i,j} K^{\text{bin}}_{ij}$. We also denote the diagonal information structures by $\mathcal{S}_{\text{d}}$, for which only the diagonal elements of the controller are allowed to be non-zero, and thus we have $K^{\text{bin}}_{ij} = 1$ iff $i = j$ and 0 otherwise. Also, for a given $i \in \{1, \cdots, n_u\}$, define $\mathtt{J}_i \triangleq \{j \in \{1, \cdots, n_y\} \mid K^{\text{bin}}_{ij} = 1\}$, which are the set of sensor measurements that control action u_i is allowed to access. For a subset $\mathtt{I} \subseteq \{1, \cdots, n_u\}$, denote its complement by $\bar{\mathtt{I}} \triangleq \{1, \cdots, n_u\} - \mathtt{I}$. Similarly define $\mathtt{J}_{\mathtt{I}} \triangleq \bigcup_{i \in \mathtt{I}} \mathtt{J}_i$. Let B_i and C_j be the i-th column of B and j-th row of C, and for any subset $\mathtt{I} = \{i_1, \cdots, i_{|\mathtt{I}|}\}$, define $B_{\mathtt{I}} \triangleq \begin{bmatrix} B_{i_1} & \cdots B_{i_{|\mathtt{I}|}} \end{bmatrix}$. Likewise, for any subset $\mathtt{J} = \{j_1, \cdots, j_{|\mathtt{J}|}\}$, define $C_{\mathtt{J}} \triangleq \begin{bmatrix} C^T_{j_1} & \cdots & C^T_{j_{|\mathtt{J}|}} \end{bmatrix}^T$. It is noteworthy to mention that for a diagonal information structure $\mathcal{S}_{\text{d}}$, we have that $\mathtt{J}_{\bar{\mathtt{I}}} = \bar{\mathtt{I}}$.

III. Review

In this section, we will first review the notion of fixed modes in Section III-A. We then review the decentralized assignability measure in Section III-B, which is the metric that we would like to approximate in this paper. Lastly, we will review a method called diagonalization in Section III-C.

A. Fixed Modes

We will briefly review the notion of fixed modes in this section. Materials in this section are from [5], [6] which are both largely derived from [1].

Definition 1: The set of fixed-modes of a plant G with respect to a sparsity pattern $\mathcal{S}$ and a type $\mathcal{T}$, is defined as:

$$\begin{aligned}\Lambda(G, \mathcal{S}, \mathcal{T}) \triangleq & \\ & \{\lambda \in \mathbb{C} \mid \lambda \in \text{eig}\left(A_{\text{CL}}(G, K)\right),\ \forall\ K \in \mathcal{S} \cap \mathcal{T}\} \\ = & \bigcap_{K \in \mathcal{S} \cap \mathcal{T}} A_{\text{CL}}(G, K),\end{aligned}$$

where $A_{\text{CL}}(G, K)$ gives the resulting closed-loop A matrix when controller K is closed around plant G.

Here the type $\mathcal{T}$ could refer to the set of static controllers which only have a non-zero feedthrough term ($\mathcal{T}^{\text{s}}$), or the set of finite dimensional LTI dynamic controllers ($\mathcal{T}^{\text{d}}$).

Remark 2: This reduces to the well-known definition of fixed modes in [1] if $\mathcal{S}$ is block-diagonal, and $\mathcal{T} = \mathcal{T}^{\text{s}}$,

An algebraic test to check for the existence of a fixed mode (similar to the PBH rank test for controllability or observability) was given in [7, Theorem 4.1]. The generalized version of this test is given as follows:

Theorem 3 ([8, Theorem 2]): Given a strictly proper plant G, and an information structure $\mathcal{S}$, we have that $\lambda \in \mathbb{C}$ is a fixed-mode of G, i.e., $\lambda \in \Lambda(G, \mathcal{S}, \mathcal{T}^{\text{s}})$, if and only if there exists a subset $\mathtt{I} \subseteq \{1, \cdots, n_u\}$ such that:

$$\text{rank} \begin{bmatrix} A - \lambda I & B_{\mathtt{I}} \\ C_{\mathtt{J}_{\bar{\mathtt{I}}}} & 0 \end{bmatrix} < n, \tag{1}$$

where n is the dimension of the state, i.e., $A \in \mathbb{R}^{n \times n}$.

B. Decentralized Assignability Measure

We will first state an existing metric on how far a system is from having decentralized fixed modes, and then review some of its properties. The materials in this section are from [2], and are adopted to the notation used in this paper.

We first define the set of plants that have the same dimension as G, and have a fixed mode with respect to $\mathcal{S}$.

Definition 4: Given the dimension of state space matrices by $\dim(G)$, and an information structure $\mathcal{S}$, define the set of *unassignable* systems as:

$$\begin{aligned}&\textbf{UNA}\left(\dim(G), \mathcal{S}\right) \triangleq \\ &\quad \{\tilde{G} \mid \tilde{G} = \left(\tilde{A}, \tilde{B}, \tilde{C}, 0\right),\ \text{where } \tilde{A} \in \mathbb{R}^{n \times n}, \tilde{B} \in \mathbb{R}^{n \times n_u}, \\ &\qquad \tilde{C} \in \mathbb{R}^{n_y \times n},\ \text{such that } \Lambda\left(\tilde{G}, \mathcal{S}, \mathcal{T}^{\text{s}}\right) \neq \varnothing\},\end{aligned} \tag{2}$$

where dependence on G is implicitly through the dimension of its state-space matrices, and dependence on $\mathcal{S}$ comes from all $\tilde{G}$ having a fixed mode with respect to $\mathcal{S}$.

We are interested in the minimum distance between G, and the set of plants that have fixed-mode(s) with respect to the information structure $\mathcal{S}$, i.e., we interested in the distance of G from $\textbf{UNA}\left(\dim(G), \mathcal{S}\right)$. To this end, define the following notion of distance:

$$\begin{aligned}&d\left(G, \textbf{UNA}\left(\dim(G), \mathcal{S}\right)\right) \triangleq \\ &\qquad \inf_{\tilde{G} \in \textbf{UNA}(\dim(G), \mathcal{S})} \left\| \begin{bmatrix} A - \tilde{A} & B - \tilde{B} \\ C - \tilde{C} & 0 \end{bmatrix} \right\|_2,\end{aligned} \tag{3}$$

where $(\tilde{A}, \tilde{B}, \tilde{C}, 0)$ is a state-space representation for $\tilde{G}$.

Vaz & Davison [2] have defined the *decentralized assignability measure* as the above distance, and have shown that it can equivalently be written as an another optimization problem:

Theorem 5 ([2, Theorem 3]): Given an LTI plant G, and an information structure $\mathcal{S}$, the *decentralized assignability*

measure is given by:

$$\sigma_{\mathrm{VD}}(G,\mathcal{S}) \triangleq d\left(G, \mathbf{UNA}\left(\dim(G),\mathcal{S}\right)\right) = \min_{\substack{\lambda\in\mathbb{C},\\ \mathtt{I}\subseteq\{1,\cdots,n_u\}}} \sigma_n\left(\begin{bmatrix} A-\lambda I & B_{\mathtt{I}} \\ C_{\mathtt{J}_{\bar{\mathtt{I}}}} & 0 \end{bmatrix}\right), \quad (4)$$

where $\mathtt{I}$ can be any non-empty proper subset, and $\mathtt{J}_{\bar{\mathtt{I}}}$ depends on $\mathcal{S}$ and $\mathtt{I}$ as stated earlier.

Remark 6: This metric is zero if and only if (1) is satisfied, which in turn is a necessary and sufficient condition for having a fixed mode.

Remark 7: This metric possesses interesting properties, but it is hard to compute due to two reasons. Firstly, minimizing over the n-th singular value is non-convex, and secondly, minimizing over the partitions $\mathtt{I}\subseteq\{1,\cdots,n_u\}$ would involve integer programming ($2^{n_u}-2$ cases). This is our main motivation to approximate (4) by easily computable methods.

C. Diagonalization

We will briefly review a technique called diagonalization. This technique could be used to transform the non-diagonal information structures ($\mathcal{S}\neq\mathcal{S}_{\mathrm{d}}$) into a diagonal one by arranging and repeating the columns of B (and rows of C) in a certain manner.

Theorem 8: Given a plant G, and an arbitrary information structure $\mathcal{S}$, let G_{d} be the *diagonalized plant* given as:

$$\begin{aligned} A_{\mathrm{d}} &= A, \qquad & B_{\mathrm{d}} &= \begin{bmatrix}(B_{\mathrm{d}})_1 & \cdots & (B_{\mathrm{d}})_{n_u}\end{bmatrix} \\ C_{\mathrm{d}} &= \begin{bmatrix}(C_{\mathrm{d}})_1^T & \cdots & (C_{\mathrm{d}})_{n_u}^T\end{bmatrix}^T, \qquad & D_{\mathrm{d}} &= 0, \end{aligned} \quad (5)$$

where $B_{\mathrm{d}}\in\mathbb{R}^{n\times a}$, $(B_{\mathrm{d}})_i\in\mathbb{R}^{n\times|\mathtt{J}_i|}$, and $(B_{\mathrm{d}})_i = [B_i \ \cdots B_i]$. Also, $C_{\mathrm{d}}\in\mathbb{R}^{a\times n}$, and $(C_{\mathrm{d}})_i = \begin{bmatrix}\cdots & C_j^T & \cdots\end{bmatrix}^T$, for all $j\in\mathtt{J}_i$. Then, we have that:

$$\Lambda(G,\mathcal{S},\mathcal{T}^{\mathrm{s}}) = \Lambda(G_{\mathrm{d}},\mathcal{S}_{\mathrm{d}},\mathcal{T}^{\mathrm{s}}). \quad (6)$$

Proof: The proof would closely follow the one of the [8, Theorem 1]. ∎

Here, dependence on $\mathcal{S}$ is implicitly through formation of $\mathtt{J}_i$. Whenever this techniques is used through this paper, we will make it clear by subscripting the state space matrices by $(\cdot)_{\mathrm{d}}$.

Remark 9: Given a plant G, and a diagonal information structure $\mathcal{S}_{\mathrm{d}}$, we have that $G_{\mathrm{d}}=G$.

IV. Relaxations of the Decentralized Assignability Measure

We are going to form the approximated version of the Vaz & Davison metric in this section. First, we will address a relaxation for the integer programming involved in (1), and study the rank constraint between the original problem and the relaxed one. Next, we will apply the aforementioned relaxation on (4) and discuss its properties. We will then utilize a convex relaxation for minimizing over the n-th singular value.

We will first propose a relaxation for the power set involved in the rank test of (1), and prove its equivalence in the following theorem. For ease of notation define:

$$F(\lambda,\alpha) \triangleq \begin{bmatrix} A-\lambda I & B_{\mathrm{d}}\,\mathrm{diag}(\alpha) \\ \mathrm{diag}(1-\alpha)\,C_{\mathrm{d}} & 0 \end{bmatrix}. \quad (7)$$

Theorem 10: Given a plant G, and an arbitrary information structure $\mathcal{S}$, let G_{d} denote the diagonalized plant as in (5). Let $\lambda\in\mathbb{C}$ be fixed. Then, there exists an $\mathtt{I}\subseteq\{1,\cdots,n_u\}$ such that (1) holds, if and only if there exists an $\alpha\in[0,1]^a$ such that:

$$\mathrm{rank}\left(F(\lambda,\alpha)\right) < n, \quad (8)$$

i.e., $\lambda\in\Lambda(G,\mathcal{S},\mathcal{T}^{\mathrm{s}})$ if and only if (8) holds for some $\alpha\in[0,1]^a$.

Proof: Proof of one direction is obvious and would only involve rendering the matrix in (8) with additional zero columns (or rows). Proof of the other direction would be achieved by observing that dropping a column (or a row), and multiplying a row (or a column) by a scalar would not increase the rank. ∎

We will use the same technique as in the above theorem to approximate (4) by:

$$\sigma_{\mathbb{B}}(G,\mathcal{S}) \triangleq \min_{\substack{\lambda\in\mathbb{C}\\ \alpha\in\mathbb{B}^a}} \sigma_n\left(F(\lambda,\alpha)\right), \quad (9)$$

and then apply the same relaxation on it to derive the following optimization problem:

$$\sigma_{\square}(G,\mathcal{S}) \triangleq \min_{\substack{\lambda\in\mathbb{C}\\ \alpha\in[0,1]^a}} \sigma_n\left(F(\lambda,\alpha)\right), \quad (10)$$

where dependence on $\mathcal{S}$ in (9) and (10) are implicitly through the diagonalization (5). The following theorem connects the above approximation to the original problem of (4), when we have a diagonal information structure $\mathcal{S}_{\mathrm{d}}$.

Theorem 11: Given a plant G, and a diagonal information structure $\mathcal{S}_{\mathrm{d}}$, we have that:

$$\sigma_{\mathrm{VD}}(G,\mathcal{S}_{\mathrm{d}}) = \sigma_{\mathbb{B}}(G,\mathcal{S}_{\mathrm{d}}) \geq \sigma_{\square}(G,\mathcal{S}_{\mathrm{d}}). \quad (11)$$

Proof: We will first prove the equality part. When $\mathcal{S}=\mathcal{S}_{\mathrm{d}}$, we have $\mathtt{J}_{\bar{\mathtt{I}}}=\bar{\mathtt{I}}$ in (4), where $\mathtt{I}\subseteq\{1,\cdots,a\}$. For $i\in\{1,\cdots,a\}$, let $\alpha_i=1$ if and only if $i\in\mathtt{I}$, and 0 otherwise. Then, given a binary α (or $\mathtt{I}$), the matrix in the RHS of (4) would be equal to $F(\lambda,\alpha)$, except for the additional zero columns (or rows). Since these extra zero columns (or rows) do not affect the existing singular values, and could only add zeros to the tail of singular values, we have that $\sigma_{\mathrm{VD}}(G,\mathcal{S}_{\mathrm{d}}) = \sigma_{\mathbb{B}}(G,\mathcal{S}_{\mathrm{d}})$. Now, to prove the inequality, observe that since $\mathbb{B}\subset[0,1]$ we have that: $\sigma_{\mathrm{VD}}(G,\mathcal{S}_{\mathrm{d}}) = \sigma_{\mathbb{B}}(G,\mathcal{S}_{\mathrm{d}}) \geq \sigma_{\square}(G,\mathcal{S}_{\mathrm{d}})$. ∎

Remark 12: When $\mathcal{S}\neq\mathcal{S}_{\mathrm{d}}$, the equivalence between $\sigma_{\mathrm{VD}}(G,\mathcal{S})$ and $\sigma_{\mathbb{B}}(G,\mathcal{S})$ does not hold anymore. However, due to Theorem 10, we have that:

$$\Lambda(G,\mathcal{S},\mathcal{T}^{\mathrm{s}})\neq\varnothing \overset{\text{Rem.6}}{\Longleftrightarrow} \sigma_{\mathrm{VD}}(G,\mathcal{S})=0, \overset{\text{Thm.10}}{\Longleftrightarrow} \sigma_{\square}(G,\mathcal{S})=0 \overset{\text{Thm.10}}{\Longleftrightarrow} \sigma_{\mathbb{B}}(G,\mathcal{S})=0$$

i.e., in the presence of a fixed mode, both of the relaxed versions (9) and (10), as well as the Vaz & Davison metric, will go to zero.

Next, we will address the optimization over the n-th singular value. There is still no efficient way to optimize over $\sigma_n(\cdot)$ in the optimization problems (9) and (10). In fact it has been shown that rank minimization is in general NP-hard [9]. However, nuclear norm is widely recognized as a convex heuristic for rank minimization [10]. There are cases that this heuristic will result in exact solutions to the rank minimization problem [9], [11], [12]. Accordingly, instead of $\sigma_n(\cdot)$, we will use this heuristic in (9) and (10) to derive the following counterparts:

$$\sigma^*_{\mathbb{B}}(G,\mathcal{S}) \triangleq \min_{\substack{\lambda\in\mathbb{C}\\ \alpha\in\mathbb{B}^a}} \|F(\lambda,\alpha)\|_*, \tag{12}$$

where $\|\cdot\|_*$ denote the nuclear norm of a matrix. We also have the following convex optimization problem:

$$\sigma^*_{\square}(G,\mathcal{S}) \triangleq \min_{\substack{\lambda\in\mathbb{C}\\ \alpha\in[0,1]^a}} \|F(\lambda,\alpha)\|_*. \tag{13}$$

Optimization problems (12) and (13) are the foundations for the computational methods proposed in the next section.

V. Computational Methods

In this section, we will propose three methods to approximate the decentralized assignability measure of Vaz & Davison. These methods will be based on the relaxations in the previous section. We will first consider using (13) directly as to suggest it as the approximated version of the original metric. Next, we will use the obtained solution from the first method to form another optimization problem that would be expected to behave more similarly to the one of the Vaz & Davison. Lastly, we will use the Alternating Direction Method of Multipliers (ADMM) to derive an iterative algorithm for (12).

Method 1 (Nuc):

Let G and an information structure $\mathcal{S}$ be given.

1) Construct the diagonalized plant G_{d} as in (5).
2) Solve the optimization problem (13) with variables $\lambda\in\mathbb{C}$, and $\alpha\in[0,1]^a$.
3) Name the obtained solution by $\lambda^{(\mathrm{M1})^\star}$ and $\alpha^{(\mathrm{M1})^\star}$, and let σ_{M1} denote the n-th singular value of the optimal matrix in (13).

The optimization problem (13) is convex, and thus could be solved with available software packages such as `cvx` toolbox [13]. However, as Theorem 11 suggests, it is desirable that α lies in its ideal binary set, i.e., $\alpha\in\mathbb{B}^a$. Enforcing $\alpha\in\mathbb{B}^a$ would result in a non-convex problem that could not be readily approached. This is the motivation to consider the following methods, in which, we will use the obtained solution from Method 1, and round the elements of the $\alpha^{(\mathrm{M1})^\star}$ to the closest binary value, and will then solve (12) again with fixed $\alpha\in\mathbb{B}^a$.

Method 2 (Nuc+Rounding):

Let G and an information structure $\mathcal{S}$ be given.

1) Construct the diagonalized plant G_{d} as in (5).
2) Apply Method 1.
3) Set $\alpha^{\mathrm{F}}\in\mathbb{B}^a$ as: $\alpha^{\mathrm{F}}\leftarrow \mathrm{round}\left(\alpha^{(\mathrm{M1})^\star}\right)$.
4) Solve the following optimization problem:
$$\min_{\lambda\in\mathbb{C}} \left\|F(\lambda,\alpha^{\mathrm{F}})\right\|_*. \tag{14}$$
5) Let σ_{M2} denote the n-th singular value of the optimal matrix in (14).

Although Method 2 will result in having a binary α, however, as the elements of α would be farther away from their ideal binary value, the rounding might not be an effective strategy, since it would not account for any other binary point in $\mathbb{B}^a$. This motivated us to consider an alternative iterative approach to directly address (12) by ADMM.

We will first derive the ADMM based algorithm for the optimization problem (12), and summarize it later in Method 3. Problem (12) can be equivalently written as:

$$\begin{aligned}&\text{minimize} \quad \|X\|_*\\ &\text{such that:} \quad X = F(\lambda,\alpha),\end{aligned}$$

with variables $X\in\mathbb{R}^{(n+a)\times(n+a)}$, $\lambda\in\mathbb{C}$, and $\alpha\in\mathbb{B}^a$. The augmented Lagrangian for this problem can be written as:

$$\begin{aligned}L_\rho(X,\lambda,\alpha,Z) = \|X\|_* + \langle Z, X-F(\lambda,\alpha)\rangle\\ + 0.5\rho\|X-F(\lambda,\alpha)\|^2_{\mathrm{F}}\end{aligned}$$

where $Z\in\mathbb{R}^{(n+a)\times(n+a)}$ is the dual variable, and $\langle L,V\rangle = \mathrm{trace}\left(L^TV\right)$. The augmented Lagrangian can be equivalently written as:

$$\begin{aligned}L_\rho(X,\lambda,\alpha,Z) = \|X\|_* + \tfrac{\rho}{2}\left\|X-F(\lambda,\alpha)+\rho^{-1}Z\right\|^2_{\mathrm{F}}\\ - (2\rho)^{-1}\|Z\|^2_{\mathrm{F}},\end{aligned} \tag{15}$$

which can be derived by expanding the terms. The ADMM consists of iteration over minimizing α, the pair (X,λ), and updating the dual variable Z. We will first derive minimization over α. To this end, partition X as:

$$X = \left[\begin{array}{c|ccc} X_A & X_{B1} & \cdots & X_{Ba}\\ \hline X_{C1} & & & \\ \vdots & & X_D & \\ X_{Ca} & & & \end{array}\right],$$

where $X_A\in\mathbb{R}^{n\times n}$, $X_{Bi}\in\mathbb{R}^{n\times 1}$, and $X_{Ci}\in\mathbb{R}^{1\times n}$. Similar partitioning also applies to Z. We have that:

$$\begin{aligned}\alpha^{(k+1)} &= \arg\min_{\alpha\in\mathbb{B}^a} L_\rho(X^{(k)},\lambda^{(k)},\alpha,Z^{(k)})\\ &= \arg\min_{\alpha\in\mathbb{B}^a}\left\|X^{(k)}-F(\lambda^{(k)},\alpha)+\rho^{-1}Z^{(k)}\right\|^2_{\mathrm{F}}\\ &\overset{\star}{=} \arg\min_{\alpha\in\mathbb{B}^a}\sum_{i=1}^{a} {f_i}^{(k)}(\alpha_i)+c,\end{aligned}$$

where the second equality follows by taking the only term of (15) that depends on α. For all $i\in\{1,\cdots,a\}$, we have defined ${f_i}^{(k)}:\mathbb{R}\to\mathbb{R}$ as:

$$\begin{aligned}{f_i}^{(k)}(x) \triangleq \left\|X^{(k)}_{Bi}+\rho^{-1}Z^{(k)}_{Bi}-(B_{\mathrm{d}})_i x\right\|^2_{\mathrm{F}}\\ + \left\|X^{(k)}_{Ci}+\rho^{-1}Z^{(k)}_{Ci}-(C_{\mathrm{d}})_i(1-x)\right\|^2_{\mathrm{F}},\end{aligned} \tag{16}$$

then, the third equality ($\stackrel{\star}{=}$) follows since different α_i appear in different rows and columns, and the Frobenius norm can be written as the square sum of the elements. Also c gathers all the terms that do not depend on α. Hence, the α minimization step separates for the individual elements α_i, and thus we can write ${\alpha_i}^{(k+1)} = \arg\min_{\alpha_i \in \mathbb{B}} {f_i}^{(k)}(\alpha_i)$, which would give the following easily checkable condition:

$$ {\alpha_i}^{(k+1)} = \begin{cases} 1 & \text{if } {f_i}^{(k)}(1) \leq {f_i}^{(k)}(0) \\ 0 & \text{otherwise.} \end{cases} \tag{17} $$

Remark 13: Since the minimization over α separates for different elements, we would only need to check the function value at $2a$ points, rather than 2^a points.

Next, we formulate the minimization over the pair (X, λ):

$$ \begin{aligned} &(X^{(k+1)}, \lambda^{(k+1)}) \\ &= \arg\min_{X,\lambda} L_\rho(X, \lambda, \alpha^{(k+1)}, Z^{(k)}) \\ &= \arg\min_{X,\lambda} \|X\|_* + 0.5\rho \left\| X - F(\lambda, \alpha^{(k+1)}) + \rho^{-1} Z^{(k)} \right\|_{\mathrm{F}}^2, \end{aligned} \tag{18} $$

with variables $X \in \mathbb{R}^{(n+a)\times(n+a)}$, and $\lambda \in \mathbb{C}$. The minimization (18) is a convex optimization problem over the pair $(X,\ \lambda)$.

Lastly, the Z-update step would be:

$$ Z^{(k+1)} = Z^{(k)} + \rho \left(X^{(k+1)} - F(\lambda^{(k+1)}, \alpha^{(k+1)}) \right). \tag{19} $$

This ADMM based algorithm is thus given as:

Method 3 (ADMM):

Let G and an information structure $\mathcal{S}$ be given.

1) Construct the diagonalized plant G_{d} as in (5).
2) We have that $X \in \mathbb{R}^{(n+a)\times(n+a)}$, $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{B}^a$, and $Z \in \mathbb{R}^{(n+a)\times(n+a)}$. Initialize $k \leftarrow 0$, and let $X^{(0)}, \lambda^{(0)}, \alpha^{(0)}, Z^{(0)}$ be all initialized to 0 as well.
3) Update $\alpha^{(k+1)}$ as (17).
4) Update $X^{(k+1)}$ and $\lambda^{(k+1)}$ as (18).
5) Update $Z^{(k+1)}$ as (19).
6) Let $\sigma_n^{(k+1)} = \sigma_n \left(F(\lambda^{(k+1)}, \alpha^{(k+1)}) \right)$.
7) **If** $\left| \sigma_n^{(k+1)} - \sigma_n^{(k)} \right| < \epsilon$, go to step 8, **else** let $k \leftarrow k+1$, and go to step 3.
8) Let $\sigma_{\mathrm{M3}} = \min_{k \geq 1} \sigma_n^{(k)}$.

The following theorem will state that the σ_{M3} obtained from Method 3 would be an upper bound for the Vaz & Davison metric, for a diagonal information structure $\mathcal{S}_{\mathrm{d}}$.

Theorem 14: Assume that a plant G, and a diagonal information structure $\mathcal{S}_{\mathrm{d}}$ are given. Apply Method 3, and let $\sigma_n^{(k)}$ and σ_{M3} be respectively given as in Step 6 and 8 of Method 3, then for all $k \geq 1$, we have that:

$$ \sigma_n^{(k)} \geq \sigma_{\mathrm{VD}}(G, \mathcal{S}_{\mathrm{d}}), \tag{20a} $$

$$ \sigma_{\mathrm{M3}} \geq \sigma_{\mathrm{VD}}(G, \mathcal{S}_{\mathrm{d}}). \tag{20b} $$

Proof: Proof is achieved by observing that at each iteration we are evaluating the objective on a feasible point, and therefore it would result in an upper bound. ■

Since the optimization problem (12) that ADMM tries to solve is non-convex, due to the binary constraints, the usual convergence results cannot be applied, and it is unlikely that global optimality will be able to be guaranteed. As one can begin to see in the next section, initial empirical results are extremely promising, showing it clearly outperforming the other algorithms and closely tracking the Vaz & Davison metric.

VI. Numerical Examples

In this section, we will provide numerical examples to compare the three proposed methods. All the plants are strictly proper, stable, and LTI, and are further centrally controllable and observable.

Example 15: Consider the following plant, with parameter $\beta \in \mathbb{R}$:

$$ A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & \beta \\ 1 & 1 \end{bmatrix}, K^{\mathrm{bin}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. $$

This plant has a fixed mode only at $\beta = 0$. We vary β and plot the n-th singular value obtained from the three proposed methods in this paper in Figure 1. The Vaz & Davison metric ($\sigma_{\mathrm{VD}}(G, \mathcal{S})$ in (4)) is computed for the numerical examples by evaluating the singular values over a discrete grid in the complex plane for each of the $2^{n_u} - 2$ possible subsets $\mathtt{I}$, which is clearly only an option for very small problems. The black curve is the Hankel based metric for decentralized settings in [3, Theorem 16]. In this example, the results of Method 1 and Method 2 collide, meaning that the α in Method 1 was already very close to its binary value. We would like to have similar behaviour to the Vaz & Davison metric, and we see that both Method 1 and Method 2 are outperformed by the ADMM algorithm of Method 3. Also, they all behave similarly near the fixed mode.

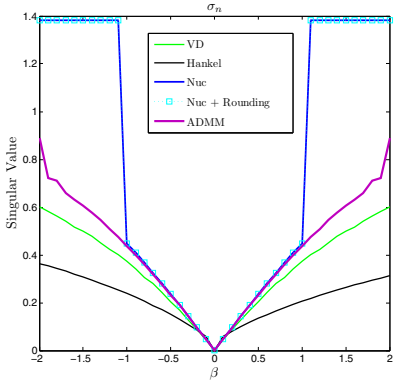

Fig. 1: Comparison of Singular Values for Example 15

Next, we give another example for which the result of Method 1 and Method 2 would be different, and would no more be similar near a fixed mode.

Example 16: Consider the plant given by:

$$ A = \mathrm{diag}(-1, -1, -1, -5), $$

$$ B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 3 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & \beta & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}, \mathcal{S} = \mathcal{S}_{\mathrm{d}}. $$

This plant has a fixed mode at $\beta = 1$. We again vary β and plot the singular value obtained from the three proposed methods, the Vaz & Davison metric, and the Hankel based metric in Figure 2. We see that Method 1 fails to detect the fixed mode, as it is non-zero at $\beta = 1$. Method 2 would detect the fixed mode, and would be close to the ideal case of Vaz & Davison near the fixed mode, but will give an unrealistic approximation as we get farther away from the fixed mode. The ADMM approach of Method 3 has the same shape as the ideal case, and closely tracks it. Also, we again see that the Hankel based metric goes to 0 only at the fixed mode, and tracks the assignability measure near the fixed mode.

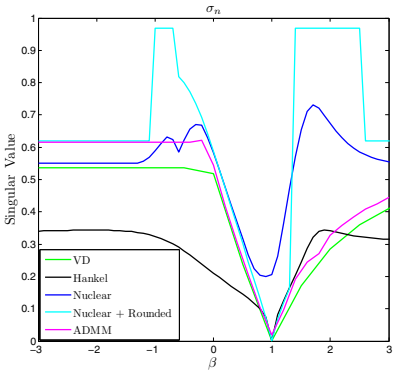

Fig. 2: Comparison of Singular Values for Example 16

Lastly, we will give an example where the Hankel based metric fails to be zero only at the presence of a fixed mode, but the ADMM approach of Method 3 would still follow the ideal case closely.

Example 17: Consider the following plant:

$$A = \operatorname{diag}(-2, -1, -1),$$

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & \beta \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, K^{\text{bin}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

This plant has a fixed mode only at $\beta = 0$. Similar to Example 15, we vary β and plot the n-th singular value obtained from the three proposed methods, the Vaz & Davison metric, and the Hankel based metric in Figure 3. The Hankel based metric is zero for all β, even when the plant has no fixed mode, and thus fails in this example. All the three proposed methods would detect the fixed mode in this case. However, we see that ADMM approach of Method 3 still has the closest behaviour to the ideal case.

VII. Conclusion

We considered the problem of how far a plant is from having a fixed mode. The decentralized assignability measure of Vaz and Davison suggests an optimization problem for computation of this measure, but it is hard to compute. We considered three methods to approximate the aforementioned measure. First, we relaxed the objective of the Vaz and Davison measure by replacing the n-the singular value with the nuclear norm, and minimizing it over the convex hull of the binary set. We then used the solution of this method to set the relaxed variable toward its closest binary value, and then solve another optimization problem. Lastly, we used the ADMM framework to derive an algorithm that would only require testing a linear number of function values over the decoupled variable. Numerical examples have shown that this method provides a close approximation of the Vaz and Davison metric across a variety of different types of fixed modes.

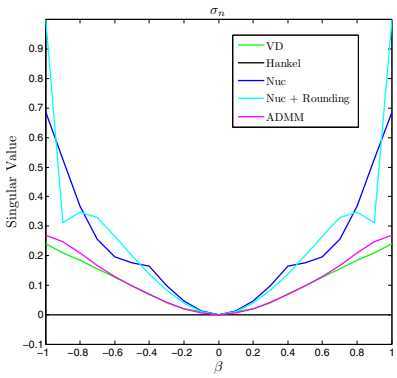

Fig. 3: Comparison of Singular Values for Example 17

References

[1] S.-H. Wang and E. J. Davison, "On the stabilization of decentralized control systems," *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 473–478, 1973.

[2] A. Vaz and E. Davison, "A measure for the decentralized assignability of eigenvalues," *Systems and Control Letters*, vol. 10, no. 3, pp. 191 – 199, 1988.

[3] A. Alavian and M. C. Rotkowitz, "On a Hankel-based measure of decentralized controllability and observability," in *Proceedings of the IFAC Workshop on Estimation and Control of Networked Systems*, 2015, pp. 227–232.

[4] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundations and Trends® in Machine Learning*, vol. 3, no. 1, pp. 1–122, 2011.

[5] A. Alavian and M. C. Rotkowitz, "Fixed modes of decentralized systems with arbitrary information structure," in *Proc. Mathematical Theory of Networks and Systems*, 2014, pp. 913–919.

[6] ——, "Stabilizing decentralized systems with arbitrary information structure," in *Proc. IEEE Conference on Decision and Control*, 2014, pp. 4032–4038.

[7] B. D. Anderson and D. J. Clements, "Algebraic characterization of fixed modes in decentralized control," *Automatica*, vol. 17, no. 5, pp. 703–712, 1981.

[8] V. Pichai, M. Sezer, and D. Šiljak, "A graph-theoretic characterization of structurally fixed modes," *Automatica*, vol. 20, no. 2, pp. 247 – 250, 1984.

[9] B. Recht, M. Fazel, and P. A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," *SIAM review*, vol. 52, no. 3, pp. 471–501, 2010.

[10] M. Fazel, H. Hindi, and S. P. Boyd, "A rank minimization heuristic with application to minimum order system approximation," in *Proc. American Control Conference*, vol. 6, 2001, pp. 4734–4739.

[11] B. Recht, W. Xu, and B. Hassibi, "Necessary and sufficient conditions for success of the nuclear norm heuristic for rank minimization," in *Proc. IEEE Conference on Decision and Control*, 2008, pp. 3065–3070.

[12] E. J. Candès and B. Recht, "Exact matrix completion via convex optimization," *Foundations of Computational mathematics*, vol. 9, no. 6, pp. 717–772, 2009.

[13] CVX Research, Inc., "CVX: Matlab software for disciplined convex programming, version 2.1," http://cvxr.com/cvx, Jun. 2014.