

On the Nearest Quadratically Invariant Information Constraint

Michael C. Rotkowitz and Nuno C. Martins

Abstract—Quadratic invariance is a condition which has been shown to allow for optimal decentralized control problems to be cast as convex optimization problems. The condition relates the constraints that the decentralization imposes on the controller to the structure of the plant. In this paper, we consider the problem of finding the closest subset and superset of the decentralization constraint which are quadratically invariant when the original problem is not. We show that this can itself be cast as a convex problem for the case where the controller is subject to delay constraints between subsystems, but that this fails when we only consider sparsity constraints on the controller. For that case, we develop an algorithm that finds the closest superset in a fixed number of steps, and discuss methods of finding a close subset.

I. INTRODUCTION

The design of decentralized controllers has been of interest for a long time, as evidenced in the surveys [1], [2], and continues to this day with the advent of complex interconnected systems. The counterexample constructed by Hans Witsenhausen in 1968 [3] clearly illustrates the fundamental reasons why problems in decentralized control are difficult.

Among the recent results in decentralized control, new approaches have been introduced that are based on algebraic principles, such as the work in [4]–[6]. Very relevant to this paper is the work in [4], [5], which classified the problems for which optimal decentralized synthesis could be cast as a convex optimization problem. Here, the plant is linear, time-invariant and it is partitioned into dynamically coupled subsystems, while the controller is also partitioned into subcontrollers. In this framework, the decentralization being imposed manifests itself as constraints on the controller to be designed, often called the *information constraint*.

The information constraint on the overall controller specifies what information is available to which controller. For instance, if information is passed between subsystems, such that each controller can access the outputs from other subsystems after different amounts of transmission time, then the information constraints are delay constraints, and may be represented by a matrix of these transmission delays. If instead, we consider each controller to be able to access the outputs from some subsystems but not from others, then the information constraint is a sparsity constraint, and may be represented by a binary matrix.

Given such pre-selected information constraints, the existence of a convex parameterization for all stabilizing controllers that satisfy the constraint can be determined via the algebraic test introduced in [4], [5], which is denoted as

quadratic invariance. In contrast with prior work, where the information constraint on the controller is fixed beforehand, this paper addresses the design of the information constraint itself. More specifically, given a plant and a pre-selected information constraint that is not quadratically invariant, we give explicit algorithms to compute the quadratically invariant information constraint that is closest to the pre-selected one. We consider finding the closest quadratically invariant superset, which corresponds to relaxing the pre-selected constraints as little as possible to get a tractable decentralized control problem, which may then be used to obtain a lower bound on the original problem, as well as finding the closest quadratically invariant subset, which corresponds to tightening the pre-selected constraints as little as possible to get a tractable decentralized control problem, which may then be used to obtain upper bounds on the original problem.

We consider the two particular cases of information constraint outlined above. In the first case, we consider constraints as transmission delays between the output of each subsystem and the subcontrollers that are connected to it. The distance between any two information constraints is quantified via a norm of the difference between the delay matrices, and we show that we can find the closest quadratically invariant set, superset, or subset as a convex optimization problem.

In the second case, we consider sparsity constraints that represent which controllers can access which subsystem outputs, and represent such constraints with binary matrices. The distance between information constraints is then given by the hamming distance, applied to the binary sparsity matrices. We provide an algorithm that gives the closest superset; that is, the quadratically invariant constraint that can be obtained by way of *allowing* the least number of additional links, and show that it terminates in a fixed number of iterations. For the problem of finding a close set or subset, we discuss some heuristic-based solutions.

Paper organization: Besides the introduction, this paper has six sections. Section II presents the notation and the basic concepts used throughout the paper. The delay and sparsity constraints adopted in our work are described in detail in Section III, while their characterization using quadratic invariance is given in Section IV. The main problems addressed in this paper are formulated and solved in Section V. Section VI briefly notes how this work also applies when assumptions of linear time-invariance are dropped, numerical examples are given in Section VII, and conclusions are given in Section VIII.

M.C. Rotkowitz is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville VIC 3010 Australia, mcrotk@unimelb.edu.au

N.C. Martins is with the Department of Electrical and Computer Engineering and the Institute for Systems Research, The University of Maryland, College Park MD 20740 USA, nmartins@umd.edu

II. PRELIMINARIES

Throughout the paper, we adopt a given causal linear time-invariant continuous-time plant P partitioned as follows:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & G \end{bmatrix}$$

Here, $P \in \mathcal{R}_p^{(n_y+n_z) \times (n_w+n_u)}$, where $\mathcal{R}_p^{q \times r}$ denotes the set of matrices of dimension q by r , whose entries are proper transfer functions of the Laplace complex variable s . Note that we abbreviate $G = P_{22}$, since we will refer to that block frequently, and so that we may refer to its subdivisions without ambiguity.

Given a causal linear time-invariant controller K in $\mathcal{R}_p^{n_u \times n_y}$, we define the **closed-loop map** by

$$f(P, K) \stackrel{\text{def}}{=} P_{11} + P_{12}K(I - GK)^{-1}P_{21}$$

where we assume that the feedback interconnection is well posed. The map $f(P, K)$ is also called the (lower) **linear fractional transformation** (LFT) of P and K . This interconnection is shown in Figure 1.

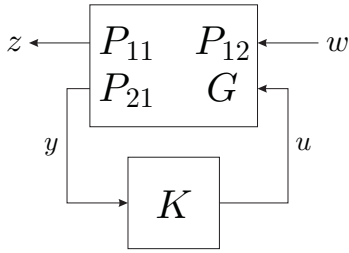


Fig. 1. Linear fractional interconnection of P and K

We suppose that there are n_y sensor measurements and n_u control actions, and thus partition the sensor measurements and control actions as

$$y = [y_1^T \ \dots \ y_{n_y}^T]^T \quad u = [u_1^T \ \dots \ u_{n_u}^T]^T$$

and then further partition G and K as

$$G = \begin{bmatrix} G_{11} & \dots & G_{1n_u} \\ \vdots & & \vdots \\ G_{n_y1} & \dots & G_{n_y n_u} \end{bmatrix} \quad K = \begin{bmatrix} K_{11} & \dots & K_{1n_y} \\ \vdots & & \vdots \\ K_{n_u1} & \dots & K_{n_u n_y} \end{bmatrix}$$

Given $A \in \mathbb{R}^{m \times n}$, we may write A in term of its columns as

$$A = [a_1 \ \dots \ a_n]$$

and then associate a vector $\text{vec}(A) \in \mathbb{R}^{mn}$ defined by

$$\text{vec}(A) \stackrel{\text{def}}{=} [a_1^T \ \dots \ a_n^T]^T$$

Further notation will be introduced as needed.

A. Delays

We define $\text{Delay}(\cdot)$ for a causal operator as the smallest amount of time in which an input can affect its output. For

any causal linear time-invariant operator $H : \mathcal{L}_e \rightarrow \mathcal{L}_e$, its delay is defined as:

$$\text{Delay}(H) \stackrel{\text{def}}{=} \inf\{\tau \geq 0 \mid z(\tau) \neq 0, z = H(w), w \in \mathcal{L}_e, \text{ where } w(t) = 0, t \leq 0\}$$

and if $H = 0$, we consider its delay to be infinite. Here $\mathcal{L}_e$ is the domain of H , which can be any extended p -normed Banach space of functions of non-negative continuous time with co-domain in the reals.

If the map H has a well defined impulse response function h , then the delay of H can be expressed as:

$$\text{Delay}(H) = \inf\{\tau \geq 0 \mid h(\tau) \neq 0\}$$

B. Sparsity

We adopt the following notation to streamline our use of sparsity patterns and sparsity constraints.

1) **Binary algebra:** Let $\mathbb{B} = \{0, 1\}$ represent the set of binary numbers. Given $x, y \in \mathbb{B}$, we define the following basic operations:

$$x + y \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } x = y = 0 \\ 1, & \text{otherwise} \end{cases}, \quad x, y \in \mathbb{B}$$

$$xy \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{otherwise} \end{cases}, \quad x, y \in \mathbb{B}$$

Given $X, Y \in \mathbb{B}^{m \times n}$, we say that $X \leq Y$ holds if and only if $X_{ij} \leq Y_{ij}$ for all i, j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$, where X_{ij} and Y_{ij} are the entries at the i -th row and j -th column of the binary matrices X and Y , respectively.

Given $X, Y, Z \in \mathbb{B}^{m \times n}$, these definitions lead to the following immediate consequences:

$$Z = X + Y \Rightarrow Z \geq X \quad (1)$$

$$X + Y = X \Leftrightarrow Y \leq X \quad (2)$$

$$X \leq Y, Y \leq X \Leftrightarrow X = Y \quad (3)$$

Given $X \in \mathbb{B}^{m \times n}$, we use the following notation to represent the total number of nonzero indices in X :

$$\mathcal{N}(X) \stackrel{\text{def}}{=} \sum_{i=1}^m \sum_{j=1}^n X_{ij}, \quad X \in \mathbb{B}^{m \times n}$$

with the sum taken in the usual way.

2) **Sparsity patterns:** Suppose that $A^{\text{bin}} \in \mathbb{B}^{m \times n}$ is a binary matrix. The following is the subspace of $\mathcal{R}_p^{m \times n}$ comprising the transfer function matrices that satisfy the sparsity constraints imposed by A^{bin} :

$$\text{Sparse}(A^{\text{bin}}) \stackrel{\text{def}}{=} \{B \in \mathcal{R}_p^{m \times n} \mid B_{ij}(j\omega) = 0 \text{ for all } i, j \text{ such that } A_{ij}^{\text{bin}} = 0 \text{ for almost all } \omega \in \mathbb{R}\}.$$

Conversely, given $B \in \mathcal{R}_p^{m \times n}$, we define $\text{Pattern}(B) \stackrel{\text{def}}{=} A^{\text{bin}}$, where A^{bin} is the binary matrix given by:

$$A_{ij}^{\text{bin}} = \begin{cases} 0, & \text{if } B_{ij}(j\omega) = 0 \text{ for almost all } \omega \in \mathbb{R} \\ 1, & \text{otherwise} \end{cases},$$

for $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$.

III. OPTIMAL CONTROL SUBJECT TO INFORMATION CONSTRAINTS

In this section, we give a detailed description of the two main types of information constraints adopted in this paper, namely, delay constraints and sparsity constraints.

A. Delay constraints

Consider a plant comprising multiple subsystems which may affect one another subject to propagation delays, and which may communicate with one another with given transmission delays. In what follows, we give a precise definition of these types of delays.

1) *Propagation Delays*: For any pair of subsystems i and j , we define the propagation delay p_{ij} as the amount of time before a controller action at subsystem j can affect an output at subsystem i , as such:

$$p_{ij} \stackrel{\text{def}}{=} \text{Delay}(G_{ij}),$$

for all $i \in \{1, \dots, n_y\}$, $j \in \{1, \dots, n_u\}$.

2) *Transmission Delays*: For any pair of subsystems k and l , we define the (total) transmission delay t_{kl} as the minimum amount of time before the controller of subsystem k may use outputs from subsystem l . Given these constraints, we can define the overall subspace of admissible controllers S such that $K \in S$ if and only if the following holds:

$$\text{Delay}(K_{kl}) \geq t_{kl},$$

for all $k \in \{1, \dots, n_u\}$, $l \in \{1, \dots, n_y\}$.

B. Sparsity Constraints

We now introduce the other main class of constraints we will consider in this paper, where each control input may access certain sensor measurements, but not others.

We represent sparsity constraints on the overall controller via a binary matrix $K^{\text{bin}} \in \mathbb{B}^{n_u \times n_y}$. Its entries can be interpreted as follows:

$$K_{kl}^{\text{bin}} = \begin{cases} 1, & \text{if control input } k \\ & \text{may access sensor measurement } l, \\ 0, & \text{if not,} \end{cases}$$

for all $k \in \{1, \dots, n_u\}$, $l \in \{1, \dots, n_y\}$.

The subspace of admissible controllers can be expressed as:

$$S = \text{Sparse}(K^{\text{bin}}).$$

From the quadratic invariance test introduced in [4], [5], we find that the relevant information about the plant is its sparsity pattern G^{bin} , obtained from:

$$G^{\text{bin}} = \text{Pattern}(G)$$

where G^{bin} is interpreted as follows:

$$G_{ij}^{\text{bin}} = \begin{cases} 1, & \text{if control input } j \\ & \text{affects sensor measurement } i, \\ 0, & \text{if not,} \end{cases}$$

for all $i \in \{1, \dots, n_y\}$, $j \in \{1, \dots, n_u\}$.

C. Optimal Control Design Via Convex Programming

Given a generalized plant P and a subspace of appropriately dimensioned causal linear time-invariant controllers S , the following is a class of constrained optimal control problems:

$$\begin{aligned} & \underset{K}{\text{minimize}} && \|f(P, K)\| \\ & \text{subject to} && K \text{ stabilizes } P \\ & && K \in S \end{aligned} \quad (4)$$

Here $\|\cdot\|$ is any norm on the closed-loop map chosen to encapsulate the control performance objectives. The delays associated with dynamics propagating from one subsystem to another, or the sparsity associated with them not propagating at all, are embedded in P . The subspace of admissible controllers, S , has been defined to encapsulate the constraints on how quickly information may be passed from one subsystem to another (delay constraints) or whether it can be passed at all (sparsity constraints). We call the subspace S the **information constraint**.

Many decentralized control problems may be expressed in the form of problem (4), including all of those addressed in [4], [7], [8]. In this paper, we focus on the case where S is defined by delay constraints or sparsity constraints as discussed above.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S . Without this constraint, the problem may be solved with many standard techniques. Note that the cost function $\|f(P, K)\|$ is in general a non-convex function of K . No computationally tractable approach is known for solving this problem for arbitrary P and S .

IV. QUADRATIC INVARIANCE

In this section, we define quadratic invariance, and we give a brief overview of related results, in particular, that if it holds then convex synthesis of optimal decentralized controllers is possible.

Definition 1: Let a causal linear time-invariant plant, represented via a transfer function matrix G in $\mathcal{R}_p^{n_y \times n_u}$, be given. If S is a subset of $\mathcal{R}_p^{n_u \times n_y}$ then S is called **quadratically invariant** under G if the following inclusion holds:

$$K G K \in S \quad \text{for all } K \in S.$$

It was shown in [4] that if S is a closed subspace and S is quadratically invariant under G , then with a change of variables, problem (4) is equivalent to the following optimization problem

$$\begin{aligned} & \underset{Q}{\text{minimize}} && \|T_1 - T_2 Q T_3\| \\ & \text{subject to} && Q \in \mathcal{RH}_\infty \\ & && Q \in S \end{aligned} \quad (5)$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$. Here $\mathcal{RH}_\infty$ is used to indicate that T_1, T_2, T_3 and Q are proper transfer function matrices with no poles in $\mathbb{C}_+$ (stable).

The optimization problem in (5) is convex. We may solve it to find the optimal Q , and then recover the optimal K for our original problem as stated in (4). If the norm of interest is the $\mathcal{H}_2$ -norm, it was shown in [4] that the problem

can be further reduced to an unconstrained optimal control problem and then solved with standard software. Similar results have been achieved [5] for function spaces beyond $\mathcal{L}_e$ as well, also showing that quadratic invariance allows optimal linear decentralized control problems to be recast as convex optimization problems.

The main focus of this paper is thus characterizing information constraints S which are as close as possible to a pre-selected one, and for which S is quadratically invariant under the plant G .

A. QI - Delay Constraints

For the case of delay constraints, it was shown in [9] that a necessary and sufficient condition for quadratic invariance is

$$t_{ki} + p_{ij} + t_{jl} \geq t_{kl}, \quad (6)$$

for all $i, l \in \{1, \dots, n_y\}$, and all $j, k \in \{1, \dots, n_u\}$.

Note that it was further shown in [9] that if we consider the typical case of n subsystems, each with its own controller, such that $n = n_y = n_u$, and if the transmission delays satisfy the triangle inequality, then the quadratic invariance test can be further reduced to the following inequality:

$$p_{ij} \geq t_{ij}, \quad (7)$$

for all $i, j \in \{1, \dots, n\}$; that is, the communication between any two nodes needs to be as fast as the propagation between the same pair of nodes.

B. QI - Sparsity Constraints

For the case of sparsity constraints, it was shown in [4] that a necessary and sufficient condition for quadratic invariance is

$$K_{ki}^{\text{bin}} G_{ij}^{\text{bin}} K_{jl}^{\text{bin}} (1 - K_{kl}^{\text{bin}}) = 0, \quad (8)$$

for all $i, l \in \{1, \dots, n_y\}$, and all $j, k \in \{1, \dots, n_u\}$.

It can be shown that this is equivalent to Condition (6) if we let

$$t_{kl} = \begin{cases} R, & \text{if } K_{kl}^{\text{bin}} = 0 \\ 0, & \text{if } K_{kl}^{\text{bin}} = 1 \end{cases}, \quad (9)$$

for all $k \in \{1, \dots, n_u\}$, and all $l \in \{1, \dots, n_y\}$, and let

$$p_{ij} = \begin{cases} R, & \text{if } G_{ij}^{\text{bin}} = 0 \\ 0, & \text{if } G_{ij}^{\text{bin}} = 1 \end{cases}, \quad (10)$$

for all $i \in \{1, \dots, n_y\}$, and all $j \in \{1, \dots, n_u\}$, for any $R > 0$, the interpretation being that a sparsity constraint can be thought of as a large delay, and a lack thereof can be thought of as no delay.

V. CLOSEST QI CONSTRAINT

We now address the main question of this paper, which is finding the closest constraints when the above conditions fail; that is, when the original problem is not quadratically invariant.

A. Closest - Delays

Suppose that we are given propagation delays $\{\tilde{p}_{ij}\}_{i=1, j=1}^{n_y, n_u}$ and transmission delays $\{\tilde{t}_{kl}\}_{k=1, l=1}^{n_u, n_y}$, and that they do not satisfy Condition (6). The problem of finding the closest constraint set, that is, the transmission delays $\{t_{kl}\}_{k=1, l=1}^{n_u, n_y}$ that are closest to $\{\tilde{t}_{kl}\}_{k=1, l=1}^{n_u, n_y}$ while satisfying (6), can be set up as follows:

$$\begin{aligned} & \underset{t}{\text{minimize}} && \|\text{vec}(t - \tilde{t})\| \\ & \text{subject to} && t_{ki} + \tilde{p}_{ij} + t_{jl} \geq t_{kl}, \quad \forall i, j, k, l, \\ & && t_{kl} \geq 0, \quad \forall k, l. \end{aligned} \quad (11)$$

This is a convex optimization problem in the new transmission delays t . The norm is arbitrary, and may be chosen to encapsulate whatever notion of closeness is most appropriate. If the 1-norm is chosen, corresponding to minimizing the sum of the differences in transmission delays, or the ∞ -norm is chosen, corresponding to minimizing the largest difference, then the problem may be cast as a linear program (LP).

If we want to find the closest quadratically invariant set, which is a superset of the original set, so that we may obtain a lower bound to the solution of the main problem (4), then we simply add the constraint $t_{kl} \leq \tilde{t}_{kl}$ for all k, l , and the problem remains convex (or remains an LP). Note that if we follow the aforementioned procedure and choose the 1-norm, then the objective is equivalent to maximizing the total delay sum $\sum_{k=1}^{n_u} \sum_{l=1}^{n_y} t_{kl}$.

Similarly, if we want to find the closest quadratically invariant set which is a subset of the original set, so that we may obtain an upper bound to the solution of the main problem (4), then we simply add the constraint $t_{kl} \geq \tilde{t}_{kl}$ for all k, l , and the problem remains convex (or remains an LP). For this procedure and if we choose the 1-norm, then the objective is equivalent to minimizing the total delay sum $\sum_{k=1}^{n_u} \sum_{l=1}^{n_y} t_{kl}$.

B. Closest - Sparsity

Now suppose that we want to construct the closest quadratically invariant set, superset, or subset, defined by sparsity constraints. We can recast a pre-selected sparsity constraint on the controller K^{bin} and a given sparsity pattern of the plant G^{bin} as in (9),(10), and then set up problem (11). The only problem is that for the resulting solution to correspond to a sparsity constraint, we need to add the binary constraints $t_{kl} \in \{0, R\}$ for all k, l , and this destroys the convexity of the problem.

1) *Sparsity Superset*: Consider first finding the closest quadratically invariant superset of the original constraint set; that is, the sparsest quadratically invariant set for which all of the original connections $y_l \rightarrow u_k$ are still in place.

This is equivalent to solving the above problem (11) with $t_{kl} \leq \tilde{t}_{kl}$ for all k, l , and with the binary constraints, an intractable combinatorial problem, but we present an algorithm which solves it and terminates in a fixed number of steps.

We can write the problem as

$$\begin{aligned} & \underset{Z \in \mathbb{B}^{n_u \times n_y}}{\text{minimize}} \quad \mathcal{N}(Z) \\ & \text{subject to} \quad ZG^{\text{bin}}Z \leq Z \\ & \quad \quad \quad K^{\text{bin}} \leq Z \end{aligned} \quad (12)$$

where additions and multiplications are as defined for the binary algebra in the preliminaries, and where we will wish to use the information constraint $S = \text{Sparse}(Z)$. The objective is defined to give us the sparsest possible solution, the first constraint ensures that the constraint set associated with the solution is quadratically invariant with respect to the plant, and the last constraint requires the resulting set of controllers to be able to access any information that could be accessed with the original constraints. Let the optimal solution to this optimization problem be denoted as $Z^* \in \mathbb{B}^{n_u \times n_y}$.

Define a sequence of sparsity constraints $\{Z_m \in \mathbb{B}^{n_u \times n_y}, m \in \mathbb{N}\}$ given by

$$Z_0 = K^{\text{bin}} \quad (13)$$

$$Z_{m+1} = Z_m + Z_m G^{\text{bin}} Z_m, \quad m \geq 0 \quad (14)$$

again using the binary algebra.

Our main result will be that this sequence converges to Z^* , and that it does so in $\log_2 n$ iterations. We first prove several preliminary lemmas, and start with a lemma elucidating which terms comprise which elements of the sequence.

Lemma 2:

$$Z_m = \sum_{s=0}^{2^m-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^s \quad \forall m \in \mathbb{N} \quad (15)$$

Proof: For $m = 0$, this follows immediately from (13). We then assume that (15) holds for a given $m \in \mathbb{N}$, and consider $m + 1$. Then,

$$\begin{aligned} Z_{m+1} &= \sum_{i=0}^{2^m-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^i + \\ &\quad \left(\sum_{k=0}^{2^m-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^k \right) G^{\text{bin}} \left(\sum_{l=0}^{2^m-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^l \right). \end{aligned}$$

All terms on the R.H.S. are of the form $K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^s$ for various $s \in \mathbb{N}$. Choosing $0 \leq i \leq 2^m - 1$ gives $0 \leq s \leq 2^m - 1$, and choosing $k = 2^m - 1$ with $0 \leq l \leq 2^m - 1$ gives $2^m \leq s \leq (2^m - 1) + 1 + (2^m - 1) = 2^{m+1} - 1$. This last term is the highest order term, so we then have $Z_{m+1} = \sum_{s=0}^{2^{m+1}-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^s$ and the proof follows by induction. ■

We now give a lemma showing how many of these terms need to be considered.

Lemma 3: The following holds for $n = \min\{n_y, n_u\}$:

$$K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^r \leq \sum_{s=0}^{n-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^s \quad \forall r \in \mathbb{N}. \quad (16)$$

Proof: Follows immediately from (1) for $r \leq n - 1$. Now consider $r \geq n$, $k \in \{1, \dots, n_u\}$, $l \in \{1, \dots, n_y\}$. Then $[K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^r]_{kl} = \sum K_{ki_1}^{\text{bin}} G_{i_1 j_1}^{\text{bin}} K_{j_1 i_2}^{\text{bin}} G_{i_2 j_2}^{\text{bin}} \dots G_{i_r j_r}^{\text{bin}} K_{j_r l}^{\text{bin}}$ where the sum is taken over all possible $i_\alpha \in \{1, \dots, n_y\}$ and

$j_\alpha \in \{1, \dots, n_u\}$. Consider an arbitrary such summand term that is equal to 1, and note that each component term must be equal to 1.

If $n = n_y$ (i), then by the pigeonhole principle either $\exists \alpha$ s.t. $i_\alpha = l$ (i.a), or $\exists \alpha, \beta$, with $\alpha \neq \beta$, s.t. $i_\alpha = i_\beta$ (i.b). In case (i.a), we have $K_{ki_1}^{\text{bin}} G_{i_1 j_1}^{\text{bin}} \dots G_{i_{\alpha-1} j_{\alpha-1}}^{\text{bin}} K_{j_{\alpha-1} l}^{\text{bin}} = 1$, or in case (i.b), we have $K_{ki_1}^{\text{bin}} \dots K_{i_\alpha j_\alpha}^{\text{bin}} G_{i_\alpha j_\alpha}^{\text{bin}} \dots K_{j_r l}^{\text{bin}} = 1$. In words, we can bypass the part of the path that merely took y_{i_α} to itself, leaving a shorter path that still connects $y_l \rightarrow u_k$.

Similarly, if $n = n_u$ (ii), then either $\exists \alpha$ s.t. $j_\alpha = k$ (ii.a), or $\exists \alpha, \beta$ with $\alpha \neq \beta$ s.t. $j_\alpha = j_\beta$ (ii.b). In case (ii.a), we have $K_{ki_1}^{\text{bin}} G_{i_1 j_1}^{\text{bin}} \dots K_{j_r l}^{\text{bin}} = 1$, or in case (ii.b), we have $K_{ki_1}^{\text{bin}} \dots G_{i_\alpha j_\alpha}^{\text{bin}} K_{j_\beta i_{\beta+1}}^{\text{bin}} \dots K_{j_r l}^{\text{bin}} = 1$, where we have now bypassed the part of the path taking u_{j_α} to itself to leave a shorter path.

We have shown that, $\forall r \geq n$, any non-zero component term of $K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^r$ has a corresponding non-zero term of strictly lower order, and the result follows. ■

We now prove another preliminary lemma showing that the optimal solution can be no more sparse than any element of the sequence.

Lemma 4: For $Z^* \in \mathbb{B}^{n_u \times n_y}$ and the sequence $\{Z_m \in \mathbb{B}^{n_u \times n_y}, m \in \mathbb{N}\}$ defined as above, the following holds:

$$Z^* \geq Z_m, \quad m \in \mathbb{N} \quad (17)$$

Proof: First, $Z^* \geq Z_0 = K^{\text{bin}}$ is given by the satisfaction of the last constraint of (12), and it just remains to show the inductive step.

Suppose that $Z^* \geq Z_m$ for some $m \in \mathbb{N}$. It then follows that

$$Z^* + Z^* G^{\text{bin}} Z^* \geq Z_m + Z_m G^{\text{bin}} Z_m.$$

From the first constraint of (12) and (2) we know that the left hand-side is just Z^* , and then using the definition of our sequence, we get $Z^* \geq Z_{m+1}$ and this completes the proof. ■

We now give a subsequent lemma, showing that if the sequence does converge, then it has converged to the optimal solution.

Lemma 5: If $Z_{m^*} = Z_{m^*+1}$ for some $m^* \in \mathbb{N}$, then $Z_{m^*} = Z^*$.

Proof: If $Z_{m^*} = Z_{m^*+1}$, then $Z_{m^*} = Z_{m^*} + Z_{m^*} G^{\text{bin}} Z_{m^*}$, and it follows from (2) that $Z_{m^*} G^{\text{bin}} Z_{m^*} \leq Z_{m^*}$. Since $Z_{m+1} \geq Z_m$ for all $m \in \mathbb{N}$, it also follows that $Z_m \geq Z_0 = K^{\text{bin}}$ for all $m \in \mathbb{N}$. Thus the two constraints of (12) are satisfied for Z_{m^*} .

Since Z^* is the sparsest binary matrix satisfying these constraints, it follows that $Z^* \leq Z_{m^*}$. Together with Lemma 4 and equation (3), it follows that $Z_{m^*} = Z^*$. ■

We now give the main result: that the sequence converges, that it does so in $\log_2 n$ steps, and that it achieves the optimal solution to our problem.

Theorem 6: The problem specified in (12) has a unique optimal solution Z^* satisfying:

$$Z_{m^*} = Z^* \quad (18)$$

where $m^* = \lceil \log_2 n \rceil$ and where $n = \min\{n_u, n_y\}$.

Proof: $Z_{m^*} = \sum_{s=0}^{2^{m^*}-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^s$ from Lemma 2, and then $Z_{m^*} = \sum_{s=0}^{n-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^s$ from Lemma 3 since $2^{m^*} \geq n$. Similarly, $Z_{m^*+1} = \sum_{s=0}^{2^{m^*+1}-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^s = \sum_{s=0}^{n-1} K^{\text{bin}} (G^{\text{bin}} K^{\text{bin}})^s$, and thus $Z_{m^*} = Z_{m^*+1}$ and the result follows from Lemma 5. ■

a) *Delay Superset (Revisited):* Suppose again that we wish to find the closest superset defined by delay constraints. This can be found by convex optimization as described in Section V-A. It can also be found with the above algorithm, where G^{bin} is replaced with a matrix of the propagation delays, K^{bin} with the given transmission delays, and where the binary algebra is replaced with the $(\min, +)$ algebra. The Z_m matrices then hold the transmission delays at each iteration, and with the appropriate inequalities flipped (since finding a superset means decreasing transmission delays), the proofs of the lemmas and the convergence theorem follow almost identically.

2) *Sparsity Subset:* We now notice an interesting asymmetry. For the case of delay constraints, if we were interested in finding the most restrictive superset (for a lower bound), or the least restrictive subset (for an upper bound), we simply flipped the sign of our final constraint, and the problem was convex either way. When we instead consider sparsity constraints, the binary constraint ruins the convexity, but we see that in the former (superset) case we can still find the closest constraint in a fixed number of iterations in polynomial time; however, for the latter (subset) case, there is no clear way to “flip” the algorithm.

This can be understood as follows. If there exist indices i, j, k, l such that $K_{ki}^{\text{bin}} = G_{ij}^{\text{bin}} = K_{jl}^{\text{bin}} = 1$, but $K_{kl}^{\text{bin}} = 0$; that is, indices for which condition (8) fails, then the above algorithm resets $K_{kl}^{\text{bin}} = 1$. In other words, if there is an indirect connection from $y_l \rightarrow u_k$, but not a direct connection, it hooks up the direct connection.

But now consider what happens if we try to develop an algorithm that goes in the other direction, that finds the least sparse constraint set which is more sparse than the original. If we again have indices for which condition (8) fails, then we need to disconnect the indirect connection, but it's not clear if we should set K_{ki}^{bin} or K_{jl}^{bin} to zero, since we could do either. The goal is, in principle, to disconnect the link that will ultimately lead to having to make the fewest subsequent disconnections, so that we end up with the closest possible constraint set to the original.

We suggest some methods for dealing with this problem. It is likely that they can be greatly improved upon, but are meant as a first cut at a reasonable polynomial time algorithm to find a close sparse subset.

For the first heuristic, we set up transmission delays and propagation delays as in (9) and (10), and then instead of adding the binary constraint and making the problem non-convex, add the relaxed constraint $0 \leq t_{kl} \leq R$ for all k, l , and solve the resulting convex problem. Then, for a set of indices violating condition (8), set K_{ki}^{bin} to zero if $t_{ki}^* \geq t_{jl}^*$, and set K_{jl}^{bin} to zero otherwise, before re-solving the convex problem. The motivation is that we disconnect the one that has a larger delay, that is, which is more constrained, in the

case where we allowed varying degrees of constraint.

The relaxed problem could instead be solved with increasing penalties on the entropy of $\text{vec}(t/R)$, to approach a binary solution, as in the study of probability collectives [10]. This method has the benefit that it could be used to find a close sparse set or subset.

For the second heuristic, we more directly keep track of how many indirect connections are associated with a direct connection. Define this weight as $w_{kl} = \sum_{i=1}^{n_y} \sum_{j=1}^{n_u} K_{ki}^{\text{bin}} G_{ij}^{\text{bin}} K_{jl}^{\text{bin}}$ thus giving the amount of 3-hop connections from $y_l \rightarrow u_k$. This is a crude measure of how many subsequent disconnections we will have to make to obtain quadratic invariance if we were to disconnect a direct path from $y_l \rightarrow u_k$. Then, given indices for which condition (8) is violated, we set K_{ki}^{bin} to zero if $w_{ki} \leq w_{jl}$, and set K_{jl}^{bin} to zero otherwise.

Note that for either heuristic, we have many options for how often to reset the guiding variables, that is, to re-solve the convex program or recalculate the weights, such as after each disconnection, or after each pass through all $n_u n_y$ indices.

It has been noticed that some of the quadratically invariant constraints for certain classes of problems, including sparsity, may be thought of as partially ordered sets [11]. This raises the possibility that work in that area, such as [12], may be leveraged to more efficiently find the closest sparse sets or subsets.

VI. NONLINEAR TIME-VARYING CONTROL

It was shown in [13] that if we consider the design of possibly nonlinear, possibly time-varying (but still causal) controllers to stabilize possibly nonlinear, possibly time-varying (but still causal) plants, then while the quadratic invariance results no longer hold, the following condition

$$K_1(I \pm GK_2) \in S \quad \text{for all } K_1, K_2 \in S$$

similarly allows for a convex parameterization of all stabilizing controllers subject to the given constraint.

This condition is equivalent to quadratic invariance when S is defined by delay constraints or by sparsity constraints, and so the algorithms in this paper may also be used to find the closest constraint for which this is achieved.

VII. NUMERICAL EXAMPLES

We present some numerical examples of the algorithms developed in this paper.

A. Example - Sparsity Constraints

We start this section by finding the closest quadratically invariant superset, with respect to the following sparsity patterns of two plants with four subsystems each ($n = 4$):

$$G_I^{\text{bin}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad G_{II}^{\text{bin}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (19)$$

The first sparsity pattern in (19) represents a plant where the first two control inputs effect not only their own subsystems, but also the subsequent subsystems, and where the last control input effects not only its own subsystem, but also the

preceding subsystem. The second sparsity pattern represents a plant where each control input effects its own subsystem and the subsequent subsystem, which also corresponds to the open daisy-chain configuration. Now consider an initial proscribed controller configuration where the controller for each subsystem has access only to the measurement from its own subsystem:

$$K^{\text{bin}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (20)$$

that is, where the controller is block diagonal. Using the algorithm specified in (13)-(14) we arrive at:

$$Z_I^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad Z_{II}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (21)$$

where Z_I^* and Z_{II}^* denote the optimal solution of (12) as applied to G_I^{bin} and G_{II}^{bin} , respectively, and thus represent the sparsity constraints of the closest quadratically invariant supersets of the set of block diagonal controllers. We see that a quadratically invariant set of controllers for the first plant (which contains all block diagonal controllers) has to have the same sparsity pattern as the plant, and an additional link from the first measurement to the third controller. We then see that any quadratically invariant set for the open daisy-chain configuration which contains the diagonal will have to be lower triangular.

B. Example - Delay Constraints

We consider $n = 4$ subsystems, with the following given propagation delays and the following proscribed transmission delays, all chosen as random uniform integers from 0 to 9:

$$\tilde{p} = \begin{bmatrix} 9 & 0 & 8 & 4 \\ 0 & 7 & 8 & 7 \\ 3 & 5 & 7 & 1 \\ 5 & 5 & 3 & 1 \end{bmatrix} \quad \tilde{t} = \begin{bmatrix} 2 & 3 & 6 & 5 \\ 5 & 2 & 2 & 9 \\ 9 & 8 & 0 & 0 \\ 7 & 9 & 8 & 5 \end{bmatrix}$$

We then display in Table I the difference between the delays of the closest quadratically invariant subset, set, and superset from the given transmission delays ($t - \tilde{t}$), as measured in the vector 1-norm, 2-norm, and ∞ -norm. These were computed by solving the convex problem of Section V-A, which was done in Matlab using the CVX package [14], and verified using `linprog()` for the 1-norm and ∞ -norm.

The delays have to be increased to reach the closest QI subsets, so the first column contains only nonnegative numbers, and the delays are decreased to get to the closest supersets, so the last column contains only nonpositive numbers, and the delays may be moved in either direction to get to the closest QI set in the middle column. Finding the closest QI superset is actually the same in any norm, as each delay between a given measurement and control action is set to the fastest indirect delay between those two signals. We indeed see that the superset is the same for the 1-norm and 2-norm. This same set of delays would also solve the problem for the ∞ -norm, but it has selected a matrix with some smaller delays, with the same maximum change of 4. This shows the problem with the lack of uniqueness that often arises when optimizing the

∞ -norm: it is indifferent to further moving the delays that have not been moved the maximum amount. Thus it should never be used to find the closest superset, and when it is appropriate to control it in finding a closest set or subset, it should be used in conjunction with another norm as well. We see that this example produces the same level of sparsity in the delay differences to closest subset and set for the 1-norm and 2-norm, though minimizing the 1-norm generally produces sparse solutions [15], and should be the norm to choose here when one wishes to alter as few of the delays as necessary.

ACKNOWLEDGMENT

The authors would like to thank Randy Cogill for useful discussions related to the delay constraints.

VIII. CONCLUSIONS

The overarching goal of this paper is the design of linear time-invariant, decentralized controllers for plants comprising dynamically coupled subsystems. Given pre-selected constraints on the controller which capture the decentralization being imposed, we addressed the question of finding the closest constraint which is quadratically invariant under the plant. Problems subject to such constraints are amenable to convex synthesis, so this is important for bounding the optimal solution to the original problem.

We focused on two particular classes of this problem. The first is where the decentralization imposed on the controller is specified by delay constraints; that is, information is passed between subsystems with some given delays, represented by a matrix of transmission delays. The second is where the decentralization imposed on the controller is specified by sparsity constraints; that is, each controller can access information from some subsystems but not others, and this is represented by a binary matrix.

For the delay constraints, we showed that finding the closest quadratically invariant constraint can be set up as a convex optimization problem. We further showed that finding the closest superset; that is, the closest set that is less restrictive than the pre-selected one, to get lower bounds on the original problem, is also a convex problem, as is finding the closest subset.

For the sparsity constraints, the convexity is lost, but we provided an algorithm which is guaranteed to give the closest quadratically invariant superset in at most $\log_2 n$ iterations, where n is the number of subsystems. We also discussed methods to give close quadratically invariant subsets.

REFERENCES

- [1] H. S. Witsenhausen, "Separation of estimation and control for discrete time systems," *Proceedings of the IEEE*, vol. 59, no. 11, pp. 1557–1566, 1971.
- [2] N. Sandell, P. Varaiya, M. Athans, and M. Safonov, "Survey of decentralized control methods for large scale systems," *IEEE Transactions on Automatic Control*, vol. 23, no. 2, pp. 108–128, February 1978.
- [3] H. S. Witsenhausen, "A counterexample in stochastic optimum control," *SIAM Journal of Control*, vol. 6, no. 1, pp. 131–147, 1968.
- [4] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 274–286, February 2006.

	Closest Subset				Closest Set				Closest Superset			
$\ \cdot\ _1$	1.74	0	0	0.57	1.77	0.08	-0.06	0.13	0	0	-2.0	0
	0	1.43	0.68	0	0	0.87	0.17	-1.0	-1.0	0	0	-2.0
	0	0	2.26	0.73	-0.65	-0.1	1.81	0.29	-4.0	-2.0	0	0
	0.27	0	0	0.32	0.06	0	0	0	0	0	0	0
$\ \cdot\ _2$	2.0	0	0	1.0	1.43	0.27	-0.27	0.65	0	0	-2.0	0
	0	1.0	0.5	0	0	0.68	0.3	-0.67	-1.0	0	0	-2.0
	0	0	2.0	0.5	-1.15	-0.3	1.42	0.02	-4.0	-2.0	0	0
	0.5	0	0	0.5	0	0	0	0.03	0	0	0	0
$\ \cdot\ _\infty$	2.0	1.25	0.56	1.29	1.33	0.64	-0.55	0.6	0	-0.65	-2.92	-0.69
	0.81	1.57	1.45	0.31	-0.05	1.07	0.9	-0.89	-2.12	-0.21	-0.32	-3.48
	0	0.36	2.0	1.48	-1.33	-0.78	1.33	0.99	-4.0	-3.44	0	0
	1.15	0.66	0.66	1.24	0.44	-0.27	-0.29	0.66	-1.07	-2.6	-2.66	-0.59

TABLE I
DISTANCE TO CLOSEST QUADRATICALLY INVARIANT DELAY CONSTRAINTS

- [5] —, “Affine controller parameterization for decentralized control over Banach spaces,” *IEEE Transactions on Automatic Control*, vol. 51, no. 9, pp. 1497–1500, September 2006.
- [6] P. G. Voulgaris, “A convex characterization of classes of problems in control with specific interaction and communication structures,” in *Proc. American Control Conference*, 2001, pp. 3128–3133.
- [7] D. D. Siljak, *Decentralized control of complex systems*. Academic Press, Boston, 1994.
- [8] X. Qi, M. Salapaka, P. Voulgaris, and M. Khammash, “Structured optimal and robust control with multiple criteria: A convex solution,” *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1623–1640, 2004.
- [9] M. Rotkowitz, R. Cogill, and S. Lall, “A simple condition for the convexity of optimal control over networks with delays,” in *Proc. IEEE Conference on Decision and Control*, 2005, pp. 6686–6691.
- [10] D. Wolpert, C. Strauss, and D. Rajnarayan, “Advances in distributed optimization using probability collectives,” *Advances in Complex Systems*, vol. 9, no. 4, pp. 383–436, 2006.
- [11] P. Shah and P. Parrilo, “A partial order approach to decentralized control,” in *Proc. IEEE Conference on Decision and Control*, 2008, pp. 4351–4356.
- [12] J. Cardinal, S. Fiorini, G. Joret, R. Jungers, and J. Munro, “An efficient algorithm for partial order production,” in *Proc. 41st annual ACM symposium on theory of computing*, 2009, pp. 93–100.
- [13] M. Rotkowitz, “Information structures preserved under nonlinear time-varying feedback,” in *Proc. American Control Conference*, 2006, pp. 4207–4212.
- [14] M. Grant and S. Boyd, “CVX: Matlab software for disciplined convex programming, version 1.21,” <http://cvxr.com/cvx>, Aug. 2010.
- [15] D. Donoho, “For most large underdetermined systems of equations, the minimal ℓ^1 -norm near-solution approximates the sparsest near-solution,” *Communications on Pure and Applied Mathematics*, vol. 59, no. 7, pp. 907–934, 2006.