Decentralized Control Information Structures Preserved Under Feedback

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Abstract

We consider the problem of constructing decentralized control systems. We formulate this problem as one of minimizing the closed-loop norm of a feedback system subject to constraints on the controller structure. We define the notion of quadratic invariance of a constraint set with respect to a system, and show that if the constraint set has this property, then the constrained minimum norm problem may be solved via convex programming. We also show that quadratic invariance is necessary and sufficient for the constraint set to be preserved under feedback.

We develop necessary and sufficient conditions under which the constraint set is quadratically invariant, and show that many examples of decentralized synthesis which have been proven to be solvable in the literature are quadratically invariant. As an example, we show that a controller which minimizes the norm of the closed-loop map may be efficiently computed in the case where distributed controllers can communicate faster than the propagation delay of the plant dynamics.

Keywords: Decentralized control, convex optimization

1 Introduction

An important problem in control is that of constructing decentralized control systems, where instead of a single controller connected to a physical system, one has multiple separate controllers, each with access to different measured information and with authority over different decision or actuation variables. Examples of such systems include automobiles on the freeway, the electricity distribution grid, flocks of aerial vehicles, and spacecraft moving in formation.

There are many variations of this problem, depending on how the limited availability of information is specified, the structure of the physical systems, and whether and how separate controllers can communicate. These variations correspond to structural features of the system and structural constraints on the allowable controllers, such as sparsity constraints. In general, finding a norm-minimizing controller subject to such constraints is not a convex optimization problem, and in many cases it is intractable. In this paper we show that if the constraints on the controller satisfy a particular property, called *quadratic invariance*, with respect to the system being controlled, then the constrained minimum-norm control problem may be reduced to a convex optimization problem.

1.1 Notation

We make use of the following notation. If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, we denote by $L(\mathcal{X},\mathcal{Y})$ the set of all bounded linear maps $A:\mathcal{X}\to\mathcal{Y}$. We abbreviate $L(\mathcal{X},\mathcal{X})$ to $L(\mathcal{X})$. A map $A\in L(\mathcal{X})$ is called ***invertible*** if there exists $B\in L(\mathcal{X})$ such that $AB=BA=I$. For $S\subset\mathcal{X}$ and $T\subset\mathcal{X}^*$ define

$$S^{\perp}=\Big\{x^*\in\mathcal{X}^*\ ;\ \langle x,x^*\rangle=0 \text{ for all } x\in S\Big\}$$
$${}^{\perp}T=\Big\{x\in\mathcal{X}\ ;\ \langle x,x^*\rangle=0 \text{ for all } x^*\in T\Big\}$$

where $\mathcal{X}^*$ is the dual-space to $\mathcal{X}$. For any map $A\in L(\mathcal{X})$ define the ***resolvent set*** $\rho(A)$ by $\rho(A)=\{\lambda\in\mathbb{C};(\lambda I-A)\text{ is invertible}\}$ and the ***resolvent*** $R_A:\rho(A)\to L(\mathcal{X})$ by $R_A(\lambda)=(\lambda I-A)^{-1}$ for all $\lambda\in\rho(A)$. We also define $\rho_{uc}(A)$ to be the unbounded connected component of $\rho(A)$.

As is standard, $\mathcal{L}_2$ is the Hilbert space of square integrable functions $f:\mathbb{R}_+\to\mathcal{X}$. For $c\geq 0$ define the delay map $D_c:\mathcal{L}_2\to\mathcal{L}_2$ by

$$y=D_cu \quad\text{if}\quad y(t)=\begin{cases}u(t-c) & \text{if } t\geq c\\ 0 & \text{otherwise}\end{cases}$$

1.2 Preliminaries

Suppose $\mathcal{U},\mathcal{W},\mathcal{Y},\mathcal{Z}$ are Banach spaces, and P is a continuous linear map $P:\mathcal{W}\times\mathcal{U}\to\mathcal{Z}\times\mathcal{Y}$. Partition P as

$$P=\begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}$$

so that $P_{11}:\mathcal{W}\to\mathcal{Z}$, $P_{12}:\mathcal{U}\to\mathcal{Z}$, $P_{21}:\mathcal{W}\to\mathcal{Y}$ and $P_{22}:\mathcal{U}\to\mathcal{Y}$. Suppose $K\in L(\mathcal{Y},\mathcal{U})$. If $I-P_{22}K$ is invertible, define $f(P,K)\in L(\mathcal{W},\mathcal{Z})$ by

$$f(P,K)=P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}$$

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The map $f(P,K)$ is called the (lower) ***linear fractional transformation*** (LFT) of P and K; we will also refer to this as the ***closed-loop map***. Given $P_{22} \in L(\mathcal{U},\mathcal{Y})$, we define the set $M \subset L(\mathcal{Y},\mathcal{U})$ of controllers K such that $f(P,K)$ is well-defined by

$$M = \Big\{ K \in L(\mathcal{Y},\mathcal{U}) \ ; \ (I - P_{22}K) \text{ is invertible} \Big\}$$

and define the subset $N \subset M$ by

$$N = \Big\{ K \in L(\mathcal{Y},\mathcal{U}) \ ; 1 \in \rho_{uc}(P_{22}K) \Big\}$$

In the remainder of the paper, we abbreviate our notation and define $G = P_{22}$.

1.3 Problem formulation

Suppose $S \subset L(\mathcal{Y},\mathcal{U})$ is a closed subspace. Given $P \in L(\mathcal{W} \times \mathcal{U}, \mathcal{Z} \times \mathcal{Y})$, we would like to solve the following problem.

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \in S \\ & K \in M \end{aligned} \tag{1}$$

Here $\|\cdot\|$ is any norm on $L(\mathcal{W},\mathcal{Z})$, chosen to encapsulate the control performance objectives, and the constraint that $K \in S$ may represent sparsity or delay constraints on K. We call the subspace S the ***information constraint***.

This problem is very general, in the sense that the signal spaces $\mathcal{U},\mathcal{W},\mathcal{Y},\mathcal{Z}$ may be continuous-time, such as $\mathcal{L}_2$, or discrete-time, such as ℓ_2, and the signals and systems may evolve over infinite time, with $\mathcal{U},\mathcal{W},\mathcal{Y},\mathcal{Z}$ function spaces over $[0,\infty)$ or $\mathbb{Z}_+$, or over finite time intervals. Also the norm on $L(\mathcal{W},\mathcal{Z})$ may represent either a deterministic measure of performance, such as the induced norm, or a stochastic measure of performance, such as the $\mathcal{H}_2$ norm.

It is also important to notice that the controller K is required to be linear. While this is a serious and non-trivial limitation, it allows us to state very sharp conditions under which the problem may be solved. Important work has also considered related problems for nonlinear control with a stochastic performance index; see Section 2.

Both P and K are required to be bounded maps. In particular, if P and K are represented by rational transfer functions, this requirement is tantamount to the constraint that P and K be stable. We relax this constraint in Section 4.1 below.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved by a simple change of variables, as discussed below. Note that the cost function $\|f(P,K)\|$ is in general a non-convex function of K. Even for finite-dimensional spaces $\mathcal{U},\mathcal{W},\mathcal{Y},\mathcal{Z}$, no computationally tractable approach is known for solving this problem for arbitrary P and S.

1.4 Quadratic invariance

The following is the major property that we will use in this paper.

Definition 1. *Suppose $G \in L(\mathcal{U},\mathcal{Y})$, and $S \subset L(\mathcal{Y},\mathcal{U})$. The set S is called **quadratically invariant** under G if*

$$KGK \in S \qquad \text{for all } K \in S$$

Note that, given G, we can define a quadratic map $\Psi : L(\mathcal{Y},\mathcal{U}) \to L(\mathcal{Y},\mathcal{U})$ by $\Psi(K) = KGK$. Then a set S is quadratically invariant if and only if S is an invariant set of Ψ; that is $\Psi(S) \subset S$.

In the following, we will show that, subject to appropriate technical assumptions on S and G, a subspace S is quadratically invariant if and only if

$$K(I - GK)^{-1} \in S \quad \iff \quad K \in S$$

It is a consequence of this result that, when S is quadratically invariant, the set of achievable closed-loop maps

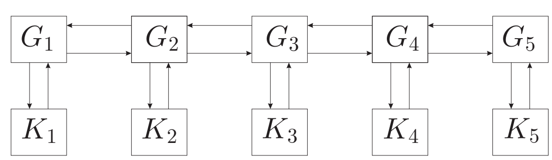
$$\Big\{ P_{11} + P_{12}K(I - GK)^{-1}P_{21} \ ; \ K \in S \Big\}$$

is affine, and hence convex.

1.5 Some examples

Many standard centralized and decentralized control problems may be represented in the form of problem (1), for specific choices of P and S. Examples include the following.

Perfectly decentralized control. We would like to design n separate controllers $\{K_1,\dots,K_n\}$, with controller K_i connected to subsystem G_i of a coupled system, as in the diagram below.



When reformulated as a synthesis problem in the LFT framework above, the constraint set S is

$$S = \Big\{ K \in L(\mathcal{Y},\mathcal{U}) \,; \, K = \operatorname{diag}(K_1,\dots,K_n) \Big\}$$

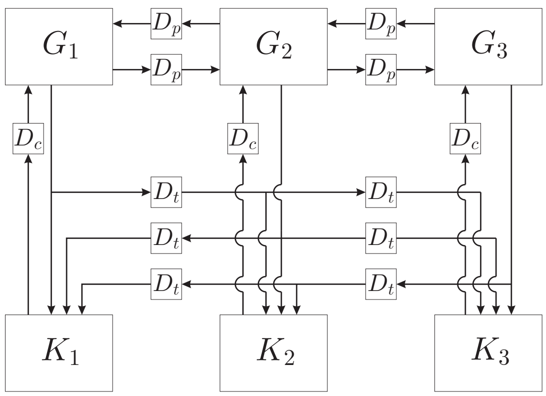
that is, S consists of those controllers that are *block-diagonal.*

Delayed measurements. In this problem, we have n linear time-invariant subsystems $\{G_1,\dots,G_n\}$, each with its respective controller K_i, arranged so that subsystem i receives signals from controller i after a computational delay of c, and controller i receives measurements from subsystem j with a transmission delay of $t|i-j|$. Also subsystem i receives signals from subsystem $i+1$ delayed by propagation delay p.

When reformulated as a synthesis problem in the LFT framework, the constraint set S may be defined as follows. Let $K \in S$ if and only if

$$K = \begin{bmatrix} D_c H_{11} & D_{t+c} H_{12} & D_{2t+c} H_{13} \\ D_{t+c} H_{21} & D_c H_{22} & D_{t+c} H_{23} \\ D_{2t+c} H_{31} & D_{t+c} H_{32} & D_c H_{33} \end{bmatrix}$$

for some linear time-invariant maps H_{ij}. The diagram below illustrates this when $n = 3$.



We will return to this example in Section 5.3, where, as an example of the utility of our approach, we provide conditions under which it may be solved via convex programming.

2 Prior work

The research in this area has a long history, and there have been many striking results which illustrate the complexity of this problem. Important early work includes that of Radner [16], who developed sufficient conditions under which minimal quadratic cost for a linear system is achieved by a linear controller. An important example was presented in 1968 by Witsenhausen [21] where it was shown that for quadratic stochastic optimal control of a linear system, subject to a decentralized information constraint called *non-classical information*, a nonlinear controller can achieve greater performance than any linear controller. An additional consequence of the work of [14, 21] is to show that under such a non-classical information pattern the cost function is no longer convex in the controller variables, a fact which today has increasing importance.

With the myth of ubiquitous linear optimality refuted, an effort began to classify the situations when it holds. In a later paper [22], Witsenhausen summarized several important results on decentralized control at that time, and gave sufficient conditions under which the problem could be reformulated so that the standard Linear-Quadratic-Gaussian (LQG) theory could be applied. Under these conditions, an optimal decentralized controller for a linear system could be chosen to be linear. Ho and Chu [11], in the framework of *team theory*, defined a more general class of information structures, called *partially nested*, for which they showed the optimal LQG controller to be linear. Roughly speaking, a plant-controller system is called partially nested if whenever the information of controller A is affected by the decision of a controller B, then A has access to all information that B has. The ideas in this paper are related to those of [11] although differ significantly in technical approach and problem formulation. Our main results determine precisely those information structures which are invariant under feedback, allowing convex synthesis of optimal linear controllers.

The computational complexity of decentralized control problems has also been extensively studied. Certain decentralized control problems, such as the static team problem of [16], have been proven to be intractable. Blondel and Tsitsiklis [3] showed that the problem of finding a stabilizing decentralized static output feedback is NP-complete. This is also the case for a discrete variant of Witsenhausen's counterexample [15]. However, there are also positive results, such as the conditions for decentralized stabilizability developed by Wang and Davison [20]. Other methods have also been developed; see for example [17].

For particular information structures, the controller optimization problem may have a tractable solution, and in particular, it was shown by Voulgaris [18] that the so-called *one-step delay information sharing pattern* problem has this property. In [10] the LEQG problem is solved in this framework, and in [18] the $\mathcal{H}_2$, $\mathcal{H}_\infty$ and $\mathcal{L}_1$ control synthesis problems are solved. A class of structured space-time systems has also been analyzed in [2], and shown to be reducible to a convex program.

Another approach to control of certain types of distributed systems has been via Fourier theory. The key idea is that, in the analysis of a spatially distributed system, it is often possible to describe the dynamics in terms of partial differential equations, or in the case of spatially discrete systems, partial difference equations. To control these systems, a theory is required which can handle multiple independent variables [5, 12]. Taking the Laplace transform with respect to time, and a Fourier transform with respect to n spatial independent variables, leads to a representation of the system as a transfer function, typically a rational function of $n + 1$ variables. Recent results in distributed control have made use of this multidimensional approach spatial invariance to give algorithms for control analysis and synthesis [1, 7, 8, 9], giving computational algorithms in terms of linear matrix inequalities.

3 Parametrization of realizable maps

In this section, we review the standard approach to solution of the feedback optimization problem (1) when the constraint that K lie in S is not present. In this case, one may use the following standard change of variables. Define the map $h : M \to L(\mathcal{Y}, \mathcal{U})$ by

$$h(K) = -K(I - GK)^{-1} \quad \text{for all } K \in M$$

We now show that h is an involution on M.

Lemma 2. *For any $G \in L(\mathcal{U},\mathcal{Y})$, the map h satisfies* $\mathrm{image}(h) = M$, *and $h : M \to M$ is a bijection, with $h \circ h = I$.*

Proof. Let $Q = h(K)$. Then a straightforward calculation shows that $(I - GQ)(I - GK) = I$, hence $\mathrm{image}(h) \subset M$. It is then immediate that $K = -Q(I - GQ)^{-1} = h(Q)$, hence $h \circ h = I$ and $\mathrm{image}(h) = M$. ■

This lemma is very useful, since we have

$$f(P,K) = P_{11} - P_{12}h(K)P_{21}$$

Hence we have the standard parametrization of all closed-loop maps which are achievable by bounded controllers K. Now we can reformulate the optimization of (1) as the following equivalent problem.

$$\begin{aligned} \text{minimize} \quad & \|P_{11} - P_{12}QP_{21}\| \\ \text{subject to} \quad & Q \in M \end{aligned} \tag{2}$$

The closed-loop map is now affine in Q, and its norm is therefore a convex function of Q. After solving this problem to find Q, one may then construct the optimal K for problem (1) via the transformation $K = h(Q)$.

This parametrization is related to the well-known *internal model principle* and Youla parametrization of stabilizing controllers. Note however that we are not considering all $Q \in L(\mathcal{Y},\mathcal{U})$, only those $Q \in M$. In many cases of practical interest M is dense in $L(\mathcal{Y},\mathcal{U})$. For transfer functions, a slightly different parametrization is appropriate; see Section 4.1 below.

Applying the above change of variables to problem (1), we arrive at the following optimization problem.

$$\begin{aligned} \text{minimize} \quad & \|P_{11} - P_{12}QP_{21}\| \\ \text{subject to} \quad & Q \in M \\ \text{subject to} \quad & h(Q) \in S \end{aligned} \tag{3}$$

The set

$$\Big\{ Q \in M \ ; \ h(Q) \in S \Big\}$$

is not convex in general, and hence this problem is not easily solved. Note that this set is equal to $h(S \cap M)$ by Lemma 2.

4 Quadratically invariant constraints under feedback

Before proving our main result, we state the following preliminary lemmas.

Lemma 3. *Suppose $G \in L(\mathcal{U},\mathcal{Y})$, and $S \subset L(\mathcal{Y},\mathcal{U})$ is a subspace. If S is a quadratically invariant under G, then*

$$K(GK)^n \in S \qquad \textit{for all } K \in S,\ n \in \mathbb{Z}_+$$

Proof. We prove this by induction. By assumption, given $K \in S$, we have that $KGK \in S$. For the induction step, assume that $K(GK)^n \in S$ for some $n \in \mathbb{Z}_+$. Then

$$\begin{aligned} 2K(GK)^{n+1} = (K + K(GK)^n)G(K + K(GK)^n) \\ - KGK - K(GK)^{2n+1} \end{aligned}$$

and since all terms on the right hand side of this equation are in S, we have $K(GK)^{n+1} \in S$. ■

Lemma 4. *Suppose $D \subset \mathbb{C}$ is an open set, $\mathcal{X}$ is a Banach space, and $q : D \to \mathcal{X}$ is analytic. Suppose that $x \in D$, and $f(y) = 0$ for all y in an open neighborhood of x. Then $f(y) = 0$ for all y in the connected component of D containing x.*

Proof. See for example Theorem 3.7 in [6]. ■

Lemma 5. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $D \subset \mathbb{C}$ is an open set, and $A : \mathcal{X} \to \mathcal{Y}$ is a bounded linear operator. Suppose $q : D \to \mathcal{X}$ is analytic, and $r : D \to \mathcal{Y}$ is given by $r = A \circ q$. Then r is analytic.*

Proof. This is a straightforward consequence of the definitions. ■

Lemma 6. *Suppose $K \in L(\mathcal{Y},\mathcal{U})$, $G \in L(\mathcal{U},\mathcal{Y})$, and $\Gamma \in L(\mathcal{Y},\mathcal{U})^*$. Define the function $q_\Gamma : \rho(GK) \to \mathbb{C}$ by*

$$q_\Gamma(\lambda) = \langle K R_{GK}(\lambda), \Gamma \rangle.$$

Then q_Γ is analytic.

Proof. Define the linear map $\gamma : L(\mathcal{Y}) \to \mathbb{C}$ by

$$\gamma(G) = \langle KG, \Gamma \rangle \quad \text{for all } G \in L(\mathcal{Y}).$$

Clearly γ is bounded, since

$$\|\gamma(G)\| \le \|K\| \|\Gamma\| \|G\| \quad \text{for all } G \in L(\mathcal{Y}).$$

Further $q_\Gamma = \gamma \circ R_{GK}$, and the resolvent is analytic, hence by Lemma 5 we have that q_Γ is analytic. ■

Main results. The following is the main result of this paper. It states that given G, if we have any constraint set S which is quadratically invariant, then the information constraints on K are equivalent to *affine constraints* on the map $Q = h(K)$.

Theorem 7. *Suppose $G \in L(\mathcal{U},\mathcal{Y})$, and $S \subset L(\mathcal{Y},\mathcal{U})$ is a closed subspace. Further suppose $N \cap S = M \cap S$. Then*

$$S \textit{ is quadratically-invariant} \iff h(S \cap M) = S \cap M$$

Proof. ($\Longrightarrow$) Suppose $K \in S \cap M$. We first show that $h(K) \in S \cap M$. For any $\Gamma \in S^\perp$ define the function $q_\Gamma : \rho(GK) \to \mathbb{C}$ by

$$q_\Gamma(\lambda) = \langle K(\lambda I - GK)^{-1}, \Gamma \rangle.$$

For any λ such that $|\lambda| > \|GK\|$, the Neumann series expansion for R_{GK} gives

$$K(\lambda I - GK)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} K(GK)^n$$

By Lemma 3 we have $K(GK)^n \in S$ for all $n \in \mathbb{Z}_+$, and hence $K(\lambda I - GK)^{-1} \in S$ since S is a closed subspace. Thus,

$$q_\Gamma(\lambda) = 0 \quad \text{for all } \lambda \text{ such that } |\lambda| > \|GK\|$$

By Lemma 6, the function q_Γ is analytic, and since $\lambda \in \rho_{uc}(GK)$ for all $|\lambda| > \|GK\|$, by Lemma 4 we have

$$q_\Gamma(\lambda) = 0 \quad \text{for all } \lambda \in \rho_{uc}(GK).$$

It follows from $K \in N$ that $1 \in \rho_{uc}(GK)$, and therefore $q_\Gamma(1) = 0$. Hence

$$\langle K(I - GK)^{-1}, \Gamma \rangle = 0 \quad \text{for all } \Gamma \in S^\perp.$$

This implies

$$K(I - GK)^{-1} \in {}^\perp(S^\perp).$$

Since S is a closed subspace, we have ${}^\perp(S^\perp) = S$ (see for example [13], p. 118) and hence we have shown $K \in S \cap M \implies h(K) \in S$. Since h is a bijective involution on M, it follows that $h(S \cap M) = S \cap M$ which was the desired result.

($\Longleftarrow$) We now turn to the converse of this result. Suppose S is not quadratically invariant. Then there exists $K_0 \in S$, such that $K_0 G K_0 \notin S$. We will construct $K \in S \cap M$ such that $h(K) \notin S \cap M$. Without loss of generality we may assume $\|K_0\| = 1$. Choose $\Gamma \in S^\perp$ with $\|\Gamma\| = 1$ such that

$$\beta = \langle K_0 G K_0, \Gamma \rangle \in \mathbb{R} \quad \text{and} \quad \beta > 0,$$

and choose $\alpha \in \mathbb{R}$ such that

$$0 < \alpha < \frac{\beta}{\|G\|\big(\beta + \|G\|\big)}$$

Let $K = \alpha K_0$. Then $\|GK\| < 1$, $K \in S \cap M$, and

$$\langle K(I - GK)^{-1}, \Gamma \rangle = \sum_{i=0}^{\infty} \langle K(GK)^i, \Gamma \rangle$$

where we have used the fact that the map γ defined in Lemma 6 is bounded. Hence

$$\begin{aligned}
\big|\langle K(I - GK)^{-1}, \Gamma \rangle\big| &= \left|\sum_{i=0}^{\infty} \langle K(GK)^i, \Gamma \rangle\right| \\
&= \left|\alpha^2 \beta + \sum_{i=2}^{\infty} \langle K(GK)^i, \Gamma \rangle\right| \\
&\geq \alpha^2 \beta - \alpha \sum_{i=2}^{\infty} \|G\|^i \alpha^i \\
&= \alpha^2 \left(\frac{\beta - \alpha\|G\|\big(\beta + \|G\|\big)}{1 - \alpha\|G\|} \right) \\
&> 0
\end{aligned}$$

Hence $K(I - GK)^{-1} \notin S$ as required. ∎

Corollary 8. *Suppose $G : \mathcal{U} \to \mathcal{Y}$ is compact and $S \subset L(\mathcal{Y}, \mathcal{U})$ is a closed subspace. Then*

$$S \textit{ is quadratically-invariant} \iff h(S \cap M) = S \cap M$$

Proof. This follows since if G is compact then GK is compact for any $K \in S$, and hence the spectrum of GK is countable, and so $N = M$. ∎

4.1 Transfer functions

Notation. We need some additional notation specifically for transfer functions. Let

$$j\mathbb{R} = \big\{ z \in \mathbb{C} \ ; \ \Re(z) = 0 \big\}$$

A rational function $G : j\mathbb{R} \to \mathbb{C}$ is called ***real-rational*** if the coefficients of its numerator and denominator polynomials are real. Similarly, a matrix-valued function $G : j\mathbb{R} \to \mathbb{C}^{m \times n}$ is called real-rational if G_{ij} is real-rational for all i, j. It is called ***proper*** if

$$\lim_{\omega \to \infty} G(j\omega) \quad \text{exists and is finite,}$$

and it is called ***strictly proper*** if

$$\lim_{\omega \to \infty} G(j\omega) = 0.$$

Let $\mathcal{R}_p^{m \times n}$ be the set of matrix-valued real-rational proper transfer functions

$$\mathcal{R}_p^{m \times n} = \Big\{ G : j\mathbb{R} \to \mathbb{C}^{m \times n} \ ; \ G \text{ proper, real-rational} \Big\}$$

and let $\mathcal{R}_{sp}^{m \times n}$ be

$$\mathcal{R}_{sp}^{m \times n} = \Big\{ G \in \mathcal{R}_p^{m \times n} \ ; \ G \text{ strictly proper} \Big\}$$

If $A \in \mathcal{R}_p^{n \times n}$ we say A is ***invertible*** if $\lim_{\omega \to \infty} A(j\omega)$ is an invertible matrix and $A(j\omega)$ is invertible for almost all $\omega \in \mathbb{R}$. Note that this is different from the definition of invertibility for the associated multiplication operator on $\mathcal{L}_2$. If A is invertible we write $B = A^{-1}$ if $B(j\omega) = A(j\omega)^{-1}$ for almost all $\omega \in \mathbb{R}$. Note that, if $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ then $I - GK$ is invertible for all $K \in \mathcal{R}_p^{n_u \times n_y}$.

Given $G \in \mathcal{R}_{sp}^{n_y \times n_u}$, we define the map $h : \mathcal{R}_p^{n_u \times n_y} \to \mathcal{R}_p^{n_y \times n_u}$ by

$$h(K) = -K(I - GK)^{-1} \quad \text{for all } K \in \mathcal{R}_p^{n_u \times n_y}$$

If $S \subset \mathcal{R}_p^{n_u \times n_y}$ is a subspace, we say S is ***frequency aligned*** if there exists a subspace $S_0 \in \mathbb{R}^{n_u \times n_y}$ such that

$$S = \Big\{ K \in \mathcal{R}_p^{n_u \times n_y} \ ; K(j\omega) \in S_0 \text{ for almost all } \omega \in \mathbb{R} \Big\}$$

As in the case when G is a linear operator, we say S is ***quadratically invariant*** under G if

$$KGK \in S \qquad \text{for all } K \in S$$

We then have the following version of Theorem 7 for transfer functions.

Theorem 9. *Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$, and $S \subset \mathcal{R}_p^{n_u \times n_y}$ is a frequency aligned subspace. Then*

$$S \text{ is quadratically invariant} \iff h(S) = S$$

Proof. The proof follows from application of Theorem 7 to the matrices $G(j\omega)$ and subspace S_0 for all $\omega \in \mathbb{R}$. ∎

4.2 Optimization over transfer functions

We now have the following equivalent problems. Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ and $S \subset \mathcal{R}_p^{n_u \times n_y}$ is a frequency aligned subspace. Then K is optimal for the problem

$$\begin{aligned} &\text{minimize} && \|f(P,K)\| \\ &\text{subject to} && K \in S \end{aligned} \tag{4}$$

if and only if $K = h(Q)$ and Q is optimal for

$$\begin{aligned} &\text{minimize} && \|P_{11} - P_{12}QP_{21}\| \\ &\text{subject to} && Q \in S \end{aligned} \tag{5}$$

This is a convex optimization problem. Solution of this problem is described in [4].

5 Examples

5.1 Sparsity constraints and computation

Many problems in decentralized control can be expressed in the form of problem (4), where S is the set of controllers that satisfy a specified sparsity constraint. In the previous section we showed that quadratic invariance of the associated subspace allowed this problem to be solved via convex optimization. In this section, we provide a computational test for quadratic invariance when the subspace S is defined by sparsity constraints. First we need a little more notation.

Suppose $A^{\text{bin}} \in \{0,1\}^{m \times n}$ is a binary matrix. We define the subspace

$$\text{Sparse}(A^{\text{bin}}) = \Big\{ B \in \mathbb{R}^{m \times n} \ ; B_{ij} = 0 \text{ for all } i,j \\ \text{such that } A_{ij} = 0 \Big\}$$

Also, if $B \in \mathbb{R}^{m \times n}$ is a matrix, let $\text{Pattern}(B)$ be the binary matrix given by

$$A^{\text{bin}} = \text{Pattern}(B) \qquad \text{if} \qquad A^{\text{bin}}_{ij} = \begin{cases} 0 & \text{if } B_{ij} = 0 \\ 1 & \text{otherwise} \end{cases}$$

The following provides the desired computational test.

Theorem 10. *Suppose $K^{\text{bin}} \in \{0,1\}^{n_u \times n_y}$, and $S = \text{Sparse}(K^{\text{bin}})$. Suppose further that $G^{\text{bin}} = \text{Pattern}(G)$. Then S is quadratically invariant under G if and only if*

$$K^{\text{bin}}_{ki}\, G^{\text{bin}}_{ij}\, K^{\text{bin}}_{jl}\, (1 - K^{\text{bin}}_{kl}) = 0$$

for all $i, l = 1, \dots, n_y$ and $j, k = 1, \dots, n_u$.

We omit the proof of the above result regarding sparsity constraints due to space constraints. This result shows that quadratic invariance can be checked in time $O(n^4)$, where $n = \max\{n_u, n_y\}$.

It is also worth noting that, if S is defined by sparsity constraints, then S is quadratically invariant under G if and only if it is quadratically invariant under all matrices with the same sparsity pattern. In general, if S is not defined by sparsity constraints, then this is not true. An example of this is when G is symmetric; then the subspace consisting of symmetric K is quadratically invariant.

5.2 Example sparsity patterns

Perfect recall. A matrix $A \in \mathbb{R}^{m \times n}$ is called a ***skyline matrix*** if for all $i = 2, \dots, m$ and all $j = 1, \dots, n$,

$$A_{i-1,j} = 0 \quad \text{if} \quad A_{i,j} = 0$$

An example is

$$K^{\text{bin}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Suppose G is lower triangular and K^{bin} is a lower triangular skyline matrix. Then $S = \text{Sparse}(K^{\text{bin}})$ is quadratically invariant under G. This case was discussed as a tractable problem in [22], where the information structure is called *perfect recall*.

The case when G and S have the same structure. It is important to notice that G and S having the same sparsity structure does not imply that S is quadratically invariant under G. For example, consider

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and let $S = \text{Sparse}(G)$. Then S is not quadratically invariant, as $G^3 \notin S$.

5.3 Distributed control with delays

We now consider the distributed control problem discussed in Section 1.5. Suppose there are n subsystems with transmission delay $t \geq 0$, propagation delay $p \geq 0$ and computational delay $c \geq 0$. When expressed in linear-fractional form, we define the allowable set of controllers is as follows. Let $K \in S$ if and only if

$$K = \begin{bmatrix} D_c H_{11} & D_{t+c} H_{12} & \dots & D_{(n-1)t+c} H_{1n} \\ D_{t+c} H_{21} & D_c H_{22} & \dots & D_{(n-2)t+c} H_{2n} \\ \vdots & & & \vdots \\ D_{(n-1)t+c} H_{n1} & \dots & & D_c H_{nn} \end{bmatrix}$$

for some $H_{ij} \in \mathcal{R}_p$ of appropriate spatial dimensions. The corresponding system G is given by

$$G = \begin{bmatrix} A_{11} & D_p A_{12} & \dots & D_{(n-1)p} A_{1n} \\ D_p A_{21} & A_{22} & \dots & D_{(n-2)p} A_{2n} \\ \vdots & & & \vdots \\ D_{(n-1)p} A_{n1} & \dots & & A_{nn} \end{bmatrix}$$

for some $A_{ij} \in \mathcal{R}_p$.

Theorem 11. *Suppose G and S are defined as above. Then if*

$$t \le p + \frac{c}{n-1}$$

then S is quadratically invariant under G.

Proof. We omit the proof due to space constraints. ∎

Other problems that have been studied in the literature and shown to be reducible to convex programs include the one-step-delayed information pattern [10], and the triangular and symmetric information patterns [19]. These problems may also be shown to be quadratically invariant.

6 Conclusions

In this paper we have developed the notion of *quadratic invariance*, and we have shown that minimum-norm control problems subject to information constraints that are quadratically invariant with respect to a plant G may be solved using convex programming. We have also shown that quadratic invariance is necessary and sufficient for the constraint set to be preserved under feedback. We have provided a computational test for quadratic invariance, and shown that many standard examples of solvable constrained optimal control problems are quadratically invariant.

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