Decentralized Control of Unstable Systems and Quadratically Invariant Information Constraints

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Abstract

We consider the problem of constructing decentralized control systems for unstable plants. We formulate this problem as one of minimizing the closed-loop norm of a feedback system subject to constraints on the controller structure, and explore which problems are amenable to convex synthesis.

For stable systems, it is known that a property called quadratic invariance of the constraint set is important. If the constraint set is quadratically invariant, then the constrained minimum-norm problem may be solved via convex programming. Examples where constraints are quadratically invariant include many classes of sparsity constraints, as well as symmetric constraints. In this paper we extend this approach to the unstable case, allowing convex synthesis of stabilizing controllers subject to quadratically invariant constraints.

Keywords: Decentralized control, convex optimization

1 Introduction

Much of conventional controls analysis assumes that the controllers to be designed all have access to the same measurements. With the advent of complex systems, decentralized control has become increasingly important, where one has multiple controllers each with access to different information. Examples of such systems include flocks of aerial vehicles, autonomous automobiles on the freeway, the power distribution grid, spacecraft moving in formation, and paper machining.

In a standard controls framework, the decentralization of the system manifests itself as sparsity or delay constraints on the controller to be designed. These constraints vary depending on the structure of the physical systems, and how separate controllers can communicate. In general, there is no known method of formulating the problem of finding a norm-minimizing controller subject to such constraints as a convex optimization problem. In many cases the problem is intractable.

In the case where the plant is stable, it has been shown that if the constraints on the controller satisfy a particular property, called *quadratic invariance*, with respect to the system being controlled, then the constrained minimum-norm control problem may be reduced to a convex optimization problem. Such quadratically invariant constraints arise in many practical contexts. In this paper we show that for unstable plants, the same condition allows us to synthesize optimal stabilizing controllers via convex programming.

1.1 Prior Work

There have been several key results regarding controller parameterization and optimization which we will extend for decentralized control. The celebrated Youla parameterization [19] showed that given a coprime factorization of the plant, one may parameterize all stabilizing controllers. The set of closed-loop maps achievable with stabilizing controllers is then affine in this parameter, an important result which converts the problem of finding the optimal stabilizing controller to a convex optimization problem, given the factorization. Zames proposed a two-step compensation scheme [20] for strongly stabilizable plants, that is, plants which can be stabilized with a stable compensator. In the first step one finds any controller which is both stable and stabilizing, and in the second one optimizes over a parameterized family of systems. This idea has been extended to nonlinear control [1], and in this paper we give conditions under which it can be extended to decentralized control.

In this paper we start with a single decentralized controller which is both stable and stabilizing, and use it to parameterize all stabilizing decentralized controllers. The resulting parameterization expresses the closed-loop system as an affine function of a stable parameter, allowing the next step, optimization of closed-loop performance, to be achieved with convex programming.

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Techniques for finding an initial stabilizing controller for decentralized systems are discussed in detail in [13], and conditions for decentralized stabilizability were developed in [16].

The problem of finding the best decentralized controller for a stable plant, intimately related to the second step, also has a long history, and there have been many striking results which illustrate the complexity of this problem. Important early work includes that of Radner [11], who developed sufficient conditions under which minimal quadratic cost for a linear system is achieved by a linear controller. An important example was presented in 1968 by Witsenhausen [17] where it was shown that for quadratic stochastic optimal control of a linear system, subject to a decentralized information constraint called *non-classical information*, a nonlinear controller can achieve greater performance than any linear controller. An additional consequence of the work of [8, 17] is to show that under such a non-classical information pattern the cost function is no longer convex in the controller variables, a fact which today has increasing importance.

With the myth of ubiquitous linear optimality refuted, an effort began to classify the situations when it holds. In a later paper [18], Witsenhausen summarized several important results on decentralized control at that time, and gave sufficient conditions under which the problem could be reformulated so that the standard Linear-Quadratic-Gaussian (LQG) theory could be applied. Under these conditions, an optimal decentralized controller for a linear system could be chosen to be linear. Ho and Chu [7], in the framework of *team theory*, defined a more general class of information structures, called *partially nested*, for which they showed the optimal LQG controller to be linear.

The computational complexity of decentralized control problems has also been extensively studied. Certain decentralized control problems, such as the static team problem of [11], have been proven to be intractable. Blondel and Tsitsiklis [3] showed that the problem of finding a stabilizing decentralized static output feedback is NP-complete. This is also the case for a discrete variant of Witsenhausen's counterexample [9].

For particular information structures, the controller optimization problem may have a tractable solution, and in particular, it was shown by Voulgaris [14] that the so-called *one-step delay information sharing pattern* problem has this property. In [5] the LEQG problem is solved in this framework, and in [14] the $\mathcal{H}_2$, $\mathcal{H}_\infty$ and $\mathcal{L}_1$ control synthesis problems are solved. A class of structured space-time systems has also been analyzed in [2], and shown to be reducible to a convex program. Several information structures are identified in [10] for which the problem of minimizing multiple objectives is reduced to a finite-dimensional convex optimization problem.

It was shown in [12] that a property called *quadratic invariance* is necessary and sufficient for the constraint set to be preserved under feedback. In the case where the plant is stable, this allows the constrained minimum-norm control problem to be reduced to a convex optimization problem. The tractable structures of [2, 5, 7, 10, 14, 15, 18] can all be shown to satisfy this property. In this paper we show that when the plant is unstable, the notion of quadratic invariance may also be used to formulate conditions under which one can synthesize optimal stabilizing controllers via convex programming.

This paper extends [12] to unstable systems and extends Section III of [20] to decentralized control.

1.2 Preliminaries

Denote by $\mathcal{R}_p^{m\times n}$ the set of matrix-valued real-rational proper transfer matrices and let $\mathcal{R}_{sp}^{m\times n}$ be the set of matrix-valued real-rational strictly proper transfer matrices.

Suppose $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, and partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

where $P_{11} \in \mathcal{R}_p^{n_z\times n_w}$. For $K \in \mathcal{R}_p^{n_u\times n_y}$ such that $I - P_{22}K$ is invertible, the ***linear fractional transformation*** (LFT) of P and K is denoted $f(P,K)$, and is defined by

$$f(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

In the remainder of the paper, we abbreviate our notation and define $G = P_{22}$. This interconnection is shown in Figure 1. We will also refer to $f(P,K)$ as the ***closed-loop map***.

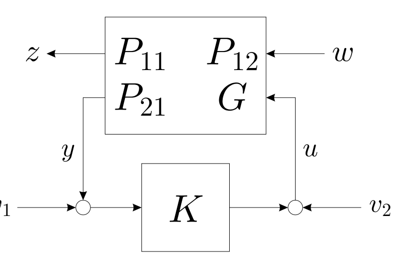

Figure 1: Linear fractional interconnection of P and K

We say that K ***stabilizes*** P if in Figure 1 the nine transfer matrices from w, v_1, v_2 to z, u, y belong to $\mathcal{RH}_\infty$. We say that K ***stabilizes*** G if in the figure the four transfer matrices from v_1, v_2 to u, y belong to $\mathcal{RH}_\infty$. P is called ***stabilizable*** if there exists $K \in \mathcal{R}_p^{n_u\times n_y}$ such that K stabilizes P, and it is called ***strongly stabilizable*** if there exists $K \in \mathcal{RH}_\infty^{n_u\times n_y}$ such that K stabilizes P. We denote by $C_{\text{stab}} \subset \mathcal{R}_p^{n_u\times n_y}$ the set of controllers $K \in \mathcal{R}_p^{n_u\times n_y}$

which stabilize P. The following standard result relates stabilization of P with stabilization of G.

Theorem 1. *Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ and $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, and suppose P is stabilizable. Then K stabilizes P if and only if K stabilizes G.*

Proof. See, for example, Chapter 4 of [6]. ∎

1.3 Problem Formulation

Suppose $S \subset \mathcal{R}_p^{n_u \times n_y}$ is a subspace. Given $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, we would like to solve the following problem

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned} \tag{1}$$

Here $\|\cdot\|$ is any norm on $\mathcal{R}_p^{n_z \times n_w}$, chosen to encapsulate the control performance objectives, and S is a subspace of admissible controllers which encapsulates the decentralized nature of the system. The norm on $\mathcal{R}_p^{n_z \times n_w}$ may be either a deterministic measure of performance, such as the induced norm, or a stochastic measure of performance, such as the $\mathcal{H}_2$ norm. Many decentralized control problems may be posed in this form. We call the subspace S the ***information constraint***.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved by a simple change of variables, as discussed below. For specific norms, the problem may also be solved using a state-space approach. Note that the cost function $\|f(P,K)\|$ is in general a non-convex function of K. No computationally tractable approach is known for solving this problem for arbitrary P and S.

1.4 Quadratic Invariance

We now turn to the main focus of this paper, which is characterizing which constraint sets S lead to tractable solutions for problem (1). In [12], a property called *quadratic invariance* was introduced for general linear operators. We define this here for the special case of transfer functions.

Definition 2. *Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$, and $S \subset \mathcal{R}_p^{n_u \times n_y}$. The set S is called* ***quadratically invariant*** *under G if*

$$KGK \in S \qquad \textit{for all } K \in S$$

Note that, given G, we can define a quadratic map $\Psi : \mathcal{R}_p^{n_u \times n_y} \to \mathcal{R}_p^{n_u \times n_y}$ by $\Psi(K) = KGK$. Then a set S is quadratically invariant if and only if S is an invariant set of Ψ; that is $\Psi(S) \subset S$. If $S \subset \mathcal{R}_p^{n_u \times n_y}$ is a subspace, we say S is ***frequency aligned*** if there exists a subspace $S_0 \in \mathbb{C}^{n_u \times n_y}$ such that

$$S = \left\{ K \in \mathcal{R}_p^{n_u \times n_y} \mid K(j\omega) \in S_0 \text{ for almost all } \omega \in \mathbb{R} \right\}$$

Given a constraint set, we define a complimentary set $S^\star \subset \mathcal{R}_p^{n_y \times n_u}$

$$S^\star = \Big\{ G \in \mathcal{R}_{sp}^{n_y \times n_u} \mid S \text{ is quadratically invariant under } G \Big\}$$

Note that the elements of $S^\star$ have dimensions which are the transpose of the elements of S, and if S is a frequency aligned subspace, then $S^\star$ is also a frequency aligned subspace.

Theorem 3. *If $S \subset \mathcal{R}_p^{n_u \times n_y}$ is a subspace, $S^\star$ is quadratically invariant under K for all $K \in S$.*

Proof. Suppose $K_1, K_2 \in S$ and $G \in S^\star$. First note that

$$\begin{aligned} &K_1GK_2 + K_2GK_1 = \\ &\qquad (K_1+K_2)G(K_1+K_2) - K_1GK_1 - K_2GK_2 \end{aligned}$$

and since all terms on the right hand side of this equation are in S, we have $K_1GK_2 + K_2GK_1 \in S$. Then we have

$$\begin{aligned} &2K_2GK_1GK_2 = \\ &\quad (K_2 + K_1GK_2 + K_2GK_1)G(K_2 + K_1GK_2 + K_2GK_1) \\ &-(K_1GK_2 + K_2GK_1)G(K_1GK_2 + K_2GK_1) - K_2GK_2 \\ &+(K_1 - K_2GK_2)G(K_1 - K_2GK_2) - K_1GK_1 \end{aligned}$$

and since all terms on the right hand side of this equation are in S, we have $K_2GK_1GK_2 \in S$ for all $K_1, K_2 \in S$ and for all $G \in S^\star$. This implies $GK_1G \in S^\star$ for all $K_1 \in S$ and for all $G \in S^\star$, and the desired result follows. ∎

This tells us that the complimentary set is quadratically invariant under any element of the constraint set, which will be very useful in proving our main result.

2 Parameterization of All Stabilizing Controllers

In this section, we review one well-known approach to solution of the feedback optimization problem (1) when the constraint that K lie in S is not present. In this case, one may use the following standard change of variables. First define the map $h : \mathcal{R}_p \times \mathcal{R}_p \to \mathcal{R}_p$ by

$$\begin{aligned} &h(G,K) = -K(I - GK)^{-1} \\ &\qquad \text{for all } G, K \text{ such that } I - GK \text{ is invertible} \end{aligned}$$

We will also make use of the notation $h_G(K) = h(G,K)$. Given $G \in \mathcal{R}_{sp}^{n_y \times n_u}$, the map h_G is an involution on $\mathcal{R}_p^{n_u \times n_y}$, as stated in the following lemma.

Lemma 4. *For any $G \in \mathcal{R}_{sp}^{n_y \times n_u}$, the map h_G satisfies* $\text{image}(h_G) = \mathcal{R}_p^{n_u \times n_y}$*, and $h_G : \mathcal{R}_p^{n_u \times n_y} \to \mathcal{R}_p^{n_u \times n_y}$ is a bijection, with $h_G \circ h_G = I$.*

Proof. A straightforward calculation shows that for any $K \in \mathcal{R}_p^{n_u \times n_y}$, $h_G(h_G(K)) = K$. It is then immediate that $\text{image}(h_G) = \mathcal{R}_p^{n_u \times n_y}$ and $h_G \circ h_G = I$. ■

For a given system P, all controllers that stabilize the system may be parameterized using the well-known Youla parameterization [19]. This parameterization is particularly simple to construct in the case where we have a nominal stabilizing controller $K_{\text{nom}} \in \mathcal{RH}_\infty$; that is, a controller that is both stable and stabilizing.

Theorem 5. *Suppose G is strictly proper, and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty$. Then all stabilizing controllers are given by*

$$C_{\text{stab}} = \Big\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \;\Big|\; Q \in \mathcal{RH}_\infty \Big\}$$

and the set of all closed-loop maps achievable with stabilizing controllers is

$$\begin{aligned} &\Big\{ f(P,K) \;\Big|\; K \in \mathcal{R}_p,\ K \text{ stabilizes } P \Big\} \\ &\qquad = \Big\{ T_1 - T_2 Q T_3 \;\Big|\; Q \in \mathcal{RH}_\infty \Big\} \end{aligned} \tag{2}$$

where

$$\begin{aligned} T_1 &= P_{11} + P_{12} K_{\text{nom}} (I - G K_{\text{nom}})^{-1} P_{21} \\ T_2 &= -P_{12} (I - K_{\text{nom}} G)^{-1} \\ T_3 &= (I - G K_{\text{nom}})^{-1} P_{21} \end{aligned} \tag{3}$$

Proof. The proof is omitted due to space constraints. ■

This theorem tells us that if the plant is strongly stabilizable, that is, if it can be stabilized by a stable controller, then given such a controller, we can parameterize the set of all stabilizing controllers. See [20] for a discussion of this, and [1] for an extension to nonlinear control. The parameterization above is very useful, since in the absence of the constraint $K \in S$, problem (1) can be reformulated as

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \end{aligned} \tag{4}$$

The closed-loop map is now affine in Q, and its norm is therefore a convex function of Q. This problem is readily solvable by, for example, the techniques in [4]. After solving this problem to find Q, one may then construct the optimal K for problem (1) via $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$.

Parameterization of all stabilizing controllers for decentralized control. We now wish to extend the above result to parameterize all stabilizing controllers $K \in \mathcal{R}_p$ that also satisfy the information constraint $K \in S$. Applying the above change of variables to problem (1), we arrive at the following optimization problem.

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \in S \end{aligned} \tag{5}$$

However, the set

$$\Big\{ Q \in \mathcal{RH}_\infty \;\Big|\; K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \in S \Big\}$$

is not convex in general, and hence this problem is not easily solved. In this paper, we develop general conditions under which this set is convex.

2.1 Quadratically Invariant Constraints

We now restate a result from [12], which is the main result of that paper applied to transfer functions.

Theorem 6. *Suppose $G \in \mathcal{R}_{sp}$ and $S \subset \mathcal{R}_p$, or $G \in \mathcal{R}_p$ and $S \subset \mathcal{R}_{sp}$, and S is a frequency aligned subspace. Then*

$$S \text{ is quadratically invariant under } G \iff h_G(S) = S$$

This will be fundamental in proving the results of this paper. The results in this section give conditions under which the set

$$\Big\{ Q \in \mathcal{RH}_\infty \;\Big|\; K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \in S \Big\}$$

is affine, and so the optimization problem (5) may be solved via convex programming. In the remainder of this section, we assume that S is a frequency aligned subspace. First we state a preliminary lemma.

Lemma 7. *Suppose $G \in \mathcal{R}_{sp}$ and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$. If S is quadratically invariant under $h(K_{\text{nom}}, G)$ then*

$$C_{\text{stab}} \cap S = \Big\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \;\Big|\; Q \in \mathcal{RH}_\infty \cap S \Big\}$$

Proof. Suppose there exists $Q \in \mathcal{RH}_\infty \cap S$ such that

$$K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big).$$

Since S is quadratically invariant under $h(K_{\text{nom}}, G)$ and S is a frequency aligned subspace, Theorem 6 implies that $h\big(h(K_{\text{nom}}, G), Q\big) \in S$, and since $K_{\text{nom}} \in S$ as well, $K \in S$. By Theorem 5, we also have $K \in C_{\text{stab}}$, so $K \in C_{\text{stab}} \cap S$.

Now suppose $K \in C_{\text{stab}} \cap S$. Let

$$Q = h\big(h(K_{\text{nom}}, G), K_{\text{nom}} - K\big).$$

We know $K_{\text{nom}} - K \in S$, and since S is quadratically invariant under $h(K_{\text{nom}}, G)$, then by Theorem 6, we also have $Q \in S$. Since h is involutive with respect to its second argument, Q is the unique element in $\mathcal{R}_p$ such that $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$, and since $K \in C_{\text{stab}}$ then by Theorem 5 we must have $Q \in \mathcal{RH}_\infty$. ■

This lemma shows that if we can find a stable $K_{\text{nom}} \in S$ which is stabilizing, and if the condition that S is quadratically invariant under $h(K_{\text{nom}}, G)$ holds, then the set of all stabilizing admissible controllers can be easily parameterized with the same change of variables from Theorem 5. We now simplify this condition.

Main result. The following theorem is the main result of this paper. It states that if the constraint set is quadratically invariant under the plant, and Q is defined as above, then the information constraints on K are equivalent to *affine constraints* on Q.

Theorem 8. *Suppose $G \in \mathcal{R}_{sp}$ and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$. If S is quadratically invariant under G then*

$$C_{\text{stab}} \cap S = \Big\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \;\big|\; Q \in \mathcal{RH}_\infty \cap S \Big\}$$

Proof. If S is quadratically invariant under G, then $G \in S^\star$. Further, by Theorem 3, $S^\star$ is quadratically invariant under K_{nom}, and then by Theorem 6, we have $h(K_{\text{nom}}, S^\star) = S^\star$. We then have $h(K_{\text{nom}}, G) \in S^\star$, and therefore S is quadratically invariant under $h(K_{\text{nom}}, G)$. By Lemma 7, this yields the desired result. ■

Remark 9. *When P is stable, we can choose $K_{\text{nom}} = 0$ and the result reduces to that analyzed in [12].*

Remark 10. *When $S = \mathcal{R}_p^{n_u \times n_y}$, which corresponds to centralized control, then the quadratic invariance condition is met and the result reduces to Theorem 5.*

2.2 Optimization Subject to Information Constraints

When the constraint set is quadratically invariant under the plant, we have the following equivalent problems. Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ and $S \subset \mathcal{R}_p^{n_u \times n_y}$ is a frequency aligned subspace. Then K is optimal for the problem

$$\begin{aligned} \text{minimize} \quad & \|f(P, K)\| \\ \text{subject to} \quad & K \text{ stabilizing} \\ & K \in S \end{aligned} \tag{6}$$

if and only if $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$ and Q is optimal for

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & Q \in S \end{aligned} \tag{7}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$ are given by equations (3). This problem may be solved via convex programming.

3 Specific Constraint Classes

In this section we show how quadratic invariance can be used for sparse and symmetric synthesis.

3.1 Sparsity Constraints

Many problems in decentralized control can be expressed in the form of problem (6), where S is the set of controllers that satisfy a specified sparsity constraint. In the previous section we showed that quadratic invariance of the associated subspace allowed this problem to be solved via convex optimization. In this section, we provide a computational test for quadratic invariance when the subspace S is defined by sparsity constraints. First we need a little more notation.

Suppose $A^{\text{bin}} \in \{0,1\}^{m \times n}$ is a binary matrix. We define the subspace

$$\begin{aligned} \text{Sparse}(A^{\text{bin}}) = \Big\{ & B \in \mathcal{R}_p^{m \times n} \;| \\ & B_{ij}(j\omega) = 0 \text{ for all } i, j \text{ such that } A^{\text{bin}}_{ij} = 0 \\ & \text{for almost all } \omega \in \mathbb{R} \Big\} \end{aligned}$$

Also, if $B \in \mathcal{R}_{sp}^{m \times n}$, let $A^{\text{bin}} = \text{Pattern}(B)$ be the binary matrix given by

$$A^{\text{bin}}_{ij} = \begin{cases} 0 & \text{if } B_{ij}(j\omega) = 0 \text{ for almost all } \omega \in \mathbb{R} \\ 1 & \text{otherwise} \end{cases}$$

3.1.1 Computational Test

The following provides a computational test for quadratic invariance when S is defined by sparsity constraints.

Theorem 11. *Suppose $S = \text{Sparse}(K^{bin})$ and $G^{bin} = \text{Pattern}(G)$ for some $K^{\text{bin}} \in \{0,1\}^{n_u \times n_y}$ and $G \in \mathcal{R}_{sp}$. Then the following are equivalent:*

(i) S is quadratically invariant under G

(ii) $K_1 G K_2 \in S \qquad$ for all $K_1, K_2 \in S$

(iii) $K^{bin}_{ki}\, G^{bin}_{ij}\, K^{bin}_{jl}\, (1 - K^{bin}_{kl}) = 0$

for all $i, l = 1, \ldots, n_y$ and $j, k = 1, \ldots, n_u$

Proof. The proof is omitted due to space constraints. ■

This result shows us several things about sparsity constraints. We see that quadratic invariance is equivalent to another condition which is stronger in general. When G is symmetric, for example, the subspace consisting of symmetric K is quadratically invariant but does not satisfy condition (*ii*). Condition (*iii*) shows that quadratic invariance can be checked in time $O(n^4)$, where $n = \max\{n_u, n_y\}$. It also shows that, if S is defined by sparsity constraints, then S is quadratically invariant under G if and only if it is quadratically invariant under all systems with the same sparsity pattern.

3.1.2 Example Sparsity Patterns

Skyline. A matrix $A \in \mathbb{R}^{m\times n}$ is called a ***skyline matrix*** if for all $i = 2, \dots, m$ and all $j = 1, \dots, n$,

$$A_{i-1,j} = 0 \quad \text{if} \quad A_{i,j} = 0$$

Examples include any of the binary matrices K^{bin} of Section 4.

Suppose $G \in \mathcal{R}_{sp}$ is lower triangular and K^{bin} is a lower triangular skyline matrix. Then

$$S = \text{Sparse}(K^{\text{bin}})$$

is quadratically invariant under G.

Other structures. Many structures which arise in practical contexts and which have been studied are in fact quadratically invariant sparsity patterns. These include nested structures, hierarchical structures, and chains [15].

3.1.3 Sparse Synthesis

The following theorem shows that for sparsity constraints, the test in Section 3.1 can be used to identify tractable decentralized control problems.

Theorem 12. *Suppose $G \in \mathcal{R}_{sp}$ and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$. Further suppose $G^{\text{bin}} = \text{Pattern}(G)$ and $S = \text{Sparse}(K^{\text{bin}})$ for some $K^{\text{bin}} \in \{0,1\}^{n_u\times n_y}$. If*

$$K_{ki}^{bin}\, G_{ij}^{bin}\, K_{jl}^{bin}\, (1 - K_{kl}^{bin}) = 0 \qquad \text{for all } i,l = 1,\dots,n_y \text{ and } j,k = 1,\dots,n_u$$

then

$$C_{\text{stab}} \cap S = \Big\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \mid Q \in \mathcal{RH}_\infty \cap S \Big\}$$

Proof. Follows immediately from Theorems 8 and 11. ■

3.2 Symmetric Synthesis

The following shows that when the plant is symmetric, the methods of this paper could be used to find the optimal symmetric stabilizing controller.

Theorem 13. *Suppose*

$$\mathbb{H}^n = \{\, A \in \mathbb{C}^{n\times n} \mid A = A^* \,\}$$

and

$$S = \{K \in \mathcal{R}_p \mid K(j\omega) \in \mathbb{H}^n \text{ for almost all } \omega \in \mathbb{R}\}.$$

Further suppose $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$ and $G \in \mathcal{R}_{sp}$ with $G(j\omega) \in \mathbb{H}^n$ for almost all $\omega \in \mathbb{R}$. Then

$$C_{\text{stab}} \cap S = \Big\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \mid Q \in \mathcal{RH}_\infty \cap S \Big\}$$

Proof. Follows immediately from Theorem 8. ■

4 Numerical Example

Consider an unstable lower triangular plant

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s-1} \end{bmatrix}$$

with P given by

$$P_{11} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \qquad P_{12} = \begin{bmatrix} G \\ I \end{bmatrix} \qquad P_{21} = \begin{bmatrix} G & I \end{bmatrix}$$

and a sequence of sparsity constraints $\{K_i^{\text{bin}}\}$

$$\begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\1&1&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\1&1&0&0&0\\1&1&0&0&1 \end{bmatrix},$$
$$\begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\1&1&0&0&0\\1&1&1&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&1&0&0&0\\1&1&1&0&0\\1&1&1&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1 \end{bmatrix}$$

defining a sequence of information constraints

$$S_i = \text{Sparse}(K_i^{\text{bin}})$$

such that each subsequent constraint is less restrictive, and such that each is quadratically invariant under G.

A stable and stabilizing controller which lies in the subspace defined by any of these sparsity constraints is given by

$$K_{\text{nom}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-4}{s+2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-4}{s+2} \end{bmatrix}$$

We can then find T_1, T_2, T_3 as in (3), and then find the stabilizing controller which minimizes the closed-loop norm subject to the sparsity constraints by solving problem (3), which is convex. Figure 2 shows the resulting minimum $\mathcal{H}_2$ norms for the six sparsity constraints as well as for a centralized controller.

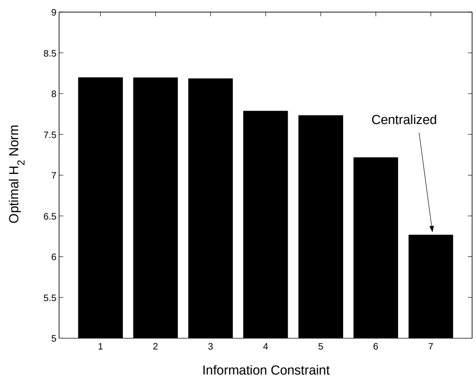

Figure 2: Optimal Norm with Information Constraints

5 Conclusions

We have shown a simple condition, called *quadratic invariance*, under which minimum-norm decentralized control problems for unstable plants may be formulated as convex optimization problems. For stable plants, it was known that this condition caused the information constraint to be invariant under feedback. We showed how to extend that result to unstable systems by constructing a parameterization of all admissible stabilizing controllers, given any controller which is both stable and stabilizing. The optimal controller may then be found with convex programming.

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