Decentralized Control Subject to Communication and Propagation Delays

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Abstract

In this paper, we prove that a wide class of distributed control problems subject to communication and propagation delays are equivalent to convex optimization problems. The results hold in both continuous and discrete time, for both stable and unstable systems. A specific example is formation flight, where each aircraft has its own controller, and the effects of an aircraft's control actions propagate to neighboring aircraft with a delay inversely proportional to the speed of sound. Here each controller may transmit sensor measurements from its aircraft to neighboring aircraft with an associated communication delay, and a consequence of these results is that if the communication delay is less than this propagation delay, then norm-optimal controllers may be found via convex programming.

1 Introduction

In this paper, we prove that a wide class of distributed control problems subject to communication and propagation delays are equivalent to convex optimization problems. The results hold in both continuous and discrete time, for both stable and unstable systems. A specific example is formation flight, where each aircraft has its own controller, and the effects of an aircraft's control actions propagate to neighboring aircraft with a delay inversely proportional to the speed of sound. Here each controller may transmit sensor measurements from its aircraft to neighboring aircraft with an associated communication delay, and a consequence of these results is that if the communication delay is less than this propagation delay, then norm-optimal controllers may be found via convex programming.

In controller optimization problems, decentralization manifests itself as delay or sparsity constraints on the controller parameters. There is no known computationally tractable method for finding the optimal controller subject to general constraints of this form, and in certain cases the problem has been proven to be intractable. We show that an important class of such problems is amenable to convex optimization, and hence solvable using standard algorithms.

1.1 Prior Work

It was shown in [8] that a property called *quadratic invariance* is necessary and sufficient for the constraint set to be preserved under feedback. In the case where the plant is stable, this allows the constrained minimum-norm control problem to be reduced to a convex optimization problem. In [9] it was shown that quadratic invariance is also sufficient when the plant is unstable. The tractable structures of [2, 5, 6, 7, 13, 14, 16] can all be shown to satisfy this property. In the case of distributed control with delays, quadratic invariance reduces to simply requiring that the transmission delay be less than the propagation delay. This is a very reasonable assumption, and this type of example represents one of the most promising applications of quadratic invariance.

The convexity of minimum-norm control problems, subject to quadratically invariant constraints, was first shown in the context of the plant and controllers being bounded linear operators, and the main result was subject to technical conditions. The result was later extended to the control of unstable systems, free from these technical conditions, provided that the constraints were *frequency-aligned*, meaning that the same constraints are imposed at each frequency. This framework is ideal for enforcing sparsity constraints. While these results are easily extended to enforce different constraints at each frequency, that is still insufficient to impose delay constraints. Even if the plant is stable and viewed as a bounded linear operator, there is also no guarantee that the constraints we need to impose will satisfy the technical conditions of the original result.

In this paper, we first present an example where a violation of these technical conditions causes the desired result to fail. Thus we elucidate that these conditions are actually necessary in general, rather than for conve-

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nience of proof. We then show that by restricting our focus to causal operators, approaching the problem from a different framework, namely extended linear spaces, and defining appropriate topologies, that we can prove a similar result free from technical conditions. We thus provide the first complete proof that synthesis of the minimum-norm controller for a distributed system with delays is a convex optimization problem if the transmission delay is less than the propagation delay.

1.2 Preliminaries

Given topological vector spaces $\mathcal{X}, \mathcal{Y}$, let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ such that T is linear and continuous. Note that if $\mathcal{X}, \mathcal{Y}$ are normed spaces, as in Theorem 5, then all such T are bounded, but that T may be unbounded in general. We abbreviate $\mathcal{L}(\mathcal{X}, \mathcal{X})$ with $\mathcal{L}(\mathcal{X})$.

Suppose $P \in \mathcal{L}(\mathcal{W} \times \mathcal{U}, \mathcal{Z} \times \mathcal{Y})$. Partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

so that $P_{11} : \mathcal{W} \to \mathcal{Z}$, $P_{12} : \mathcal{U} \to \mathcal{Z}$, $P_{21} : \mathcal{W} \to \mathcal{Y}$ and $P_{22} : \mathcal{U} \to \mathcal{Y}$. Suppose $K \in \mathcal{L}(\mathcal{Y}, \mathcal{U})$. If $I - P_{22}K$ is invertible, define $f(P, K) \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ by

$$f(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

The map $f(P, K)$ is called the (lower) ***linear fractional transformation*** (LFT) of P and K; we will also refer to this as the ***closed-loop map***. In the remainder of the paper, we abbreviate our notation and define $G = P_{22}$.

Denote by $\mathcal{R}_p^{m \times n}$ the set of matrix-valued real-rational proper transfer matrices and let $\mathcal{R}_{sp}^{m \times n}$ be the set of matrix-valued real-rational strictly proper transfer matrices. We denote by $C_{\text{stab}} \subseteq \mathcal{R}_p^{n_u \times n_y}$ the set of controllers $K \in \mathcal{R}_p^{n_u \times n_y}$ which stabilize P.

Banach spaces. When $\mathcal{U}, \mathcal{Y}$ are Banach spaces, we also use the following notation.

Given $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, we define the set $M \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$ of controllers K such that $f(P, K)$ is well-defined by

$$M = \Big\{ K \in \mathcal{L}(\mathcal{Y}, \mathcal{U}) \;\Big|\; (I - GK) \text{ is invertible} \Big\}$$

For any Banach space $\mathcal{X}$ and bounded linear operator $A \in \mathcal{L}(\mathcal{X})$ define the ***resolvent set*** $\rho(A)$ by $\rho(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is invertible}\}$ and the ***resolvent*** $R_A : \rho(A) \to \mathcal{L}(\mathcal{X})$ by $R_A(\lambda) = (\lambda I - A)^{-1}$ for all $\lambda \in \rho(A)$. We also define $\rho_{uc}(A)$ to be the unbounded connected component of $\rho(A)$.

Note that $1 \in \rho(GK)$ for all $K \in M$, and define the subset $N \subseteq M$ by

$$N = \Big\{ K \in \mathcal{L}(\mathcal{Y}, \mathcal{U}) \;\Big|\; 1 \in \rho_{uc}(GK) \Big\}$$

Topology. Let $\mathcal{X}$ be a vector space and $\{\|\cdot\|_\alpha \mid \alpha \in I\}$ be a family of semi-norms on $\mathcal{X}$. The family is called ***sufficient*** if for all $x \in \mathcal{X}$ such that $x \neq 0$ there exists $\alpha \in I$ such that $\|x\|_\alpha \neq 0$. The topology generated by all open $\|\cdot\|_\alpha$-balls is called the topology generated by the family of semi-norms. If the family is sufficient, convergence in this topology is equivalent to convergence in every semi-norm, and continuity of a linear operator is equivalent to continuity in every semi-norm. See, for example, [19, 11].

Extended spaces. We introduce some new notation for extended linear spaces. These spaces are utilized extensively in [4, 15].

We define the truncation operator P_T for all $T \in \mathbb{R}_+$ on all functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f_T = P_T f$ is given by

$$f_T(t) = \begin{cases} f(t) & \text{if } t \leq T \\ 0 & \text{if } t > T \end{cases}$$

and hereafter abbreviate $P_T f$ as f_T. We make use of the standard L_p Banach spaces equipped with the usual p-norm, and the extended spaces

$$L_{pe} = \{ f : \mathbb{R}_+ \to \mathbb{R} \mid f_T \in L_p \text{ for all } T \in \mathbb{R}_+ \} \qquad \text{for all } p \geq 1$$

We let the topology on L_{2e} be generated by the sufficient family of semi-norms $\{\|\cdot\|_T \mid T \in \mathbb{R}_+\}$ where $\|f\|_T = \|P_T f\|_{L_2}$, and let the topology on $\mathcal{L}(L_{2e}^m, L_{2e}^n)$ be generated by the sufficient family of semi-norms $\{\|\cdot\|_T \mid T \in \mathbb{R}_+\}$ where $\|A\|_T = \|P_T A\|_{L_2^m \to L_2^n}$

We use similar notation for discrete time. As is standard, we extend the discrete-time Banach spaces ℓ_p to the extended space

$$\ell_e = \{ f : \mathbb{Z}_+ \to \mathbb{R} \mid f_T \in \ell_\infty \text{ for all } T \in \mathbb{Z}_+ \}$$

Note that in discrete time, all extended spaces contain the same elements, since the common requirement is that the sequence is finite at any finite index. This motivates the abbreviated notation of ℓ_e.

We let the topology on ℓ_e be generated by the sufficient family of semi-norms $\{\|\cdot\|_T \mid T \in \mathbb{Z}_+\}$ where $\|f\|_T = \|P_T f\|_{\ell_2}$, and let the topology on $\mathcal{L}(\ell_e^m, \ell_e^n)$ be generated by the sufficient family of semi-norms $\{\|\cdot\|_T \mid T \in \mathbb{Z}_+\}$ where $\|A\|_T = \|P_T A\|_{\ell_2^m \to \ell_2^n}$.

When the dimensions are implied by context, we omit the superscripts of $\mathcal{R}_p^{m \times n}, \mathcal{R}_{sp}^{m \times n}, \mathcal{RH}_\infty^{m \times n}, L_{pe}^{m \times n}, \ell_e^{m \times n}$. We will indicate the restriction of an operator A to $L_2[0, T]$ or $0, \ldots, T$ by $A|_T$, and the restriction and truncation of an operator as $A_T = P_T A|_T$. Thus for every semi-norm in this paper, one may write $\|A\|_T = \|A_T\|$. Given a set S, we also denote $S_T = \{P_T A|_T \;;\; A \in S\}$.

2 Problem Formulation

Suppose S is a subspace of the vector space of controllers under consideration. An example would be $S \subseteq \mathcal{R}_p^{n_u \times n_y}$, although in this paper we will also consider other possible spaces of admissible controllers. Given P we would like to solve the following problem

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned} \tag{1}$$

Here $\|\cdot\|$ is any norm chosen to encapsulate the control performance objectives, and S is a subspace of admissible controllers which encapsulates the decentralized nature of the system. Many decentralized control problems may be posed in this form. We call the subspace S the ***information constraint***.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved by a simple change of variables, as discussed below. Note that the cost function $\|f(P,K)\|$ is in general a non-convex function of K. No computationally tractable approach is known for solving this problem for arbitrary P and S.

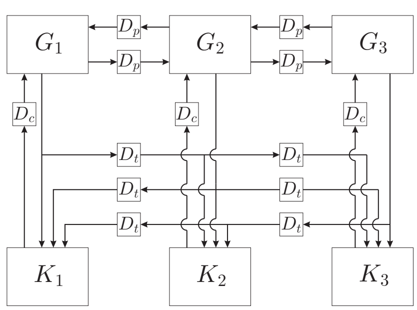

Figure 1: Optimal Norm with Delay Constraints

2.1 Motivating Example

The main motivating mathematical problem for this paper is illustrated in Figure 1. In this problem, we have n linear time-invariant causal subsystems $\{G_1, \dots, G_n\}$, each with its respective controller K_i, arranged so that subsystem i receives signals from controller i after a computational delay of c, controller i receives measurements from subsystem j with a transmission delay of $t|i-j|$, and subsystem i receives signals from subsystem j after a propagation delay $p|i-j|$.

This problem can be written in the form of (1), where S is defined as follows. Let D_τ represent a delay operator of time τ. Then $K \in S$ if and only if

$$K = \begin{bmatrix} D_c H_{11} & D_{t+c} H_{12} & D_{2t+c} H_{13} \\ D_{t+c} H_{21} & D_c H_{22} & D_{t+c} H_{23} \\ D_{2t+c} H_{31} & D_{t+c} H_{32} & D_c H_{33} \end{bmatrix}$$

for some linear time-invariant maps H_{ij}.

It was observed in [8] that the above constraint set S is quadratically invariant if

$$t \le p + \frac{c}{(n-1)}.$$

In the absence of computational delay, this condition reduces to simply requiring the transmission delay to be less than or equal to the propagation delay. This property holds for the formation flight example when the controllers can transmit their information faster than the speed of sound. In the presence of computational delay, we see that the condition is surprisingly relaxed. In this paper we provide proof that this problem, along with a wide class of others, may be solved with convex optimization.

2.2 Parameterization of All Stabilizing Controllers

In this section, we review one well-known approach to solution of the feedback optimization problem (1) when the constraint that K lie in S is not present. In this case, one may use the following standard change of variables. First define the map h as

$$h(G,K) = -K(I-GK)^{-1}$$
$$\text{for all } G, K \text{ such that } I - GK \text{ is invertible}$$

We will also make use of the notation $h_G(K) = h(G,K)$. Given G, the map h_G is an involution on M, as shown in [8].

For a given system P, all controllers that stabilize the system may be parameterized using the well-known Youla parameterization [17]. This parameterization is particularly simple to construct in the case where we have a nominal stabilizing controller $K_{\text{nom}} \in \mathcal{RH}_\infty$; that is, a controller that is both stable and stabilizing.

Theorem 1. *Suppose G is strictly proper, and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty$. Then all stabilizing controllers are given by*

$$C_{\text{stab}} = \Big\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \;\Big|\; Q \in \mathcal{RH}_\infty \Big\}$$

and the set of all closed-loop maps achievable with stabilizing controllers is

$$\begin{aligned} \Big\{ f(P,K) \;\Big|\; K \in \mathcal{R}_p,\ K \text{ stabilizes } P \Big\} \\ = \Big\{ T_1 - T_2 Q T_3 \;\Big|\; Q \in \mathcal{RH}_\infty \Big\} \end{aligned} \tag{2}$$

where

$$\begin{aligned} T_1 &= P_{11} + P_{12}K_{\text{nom}}(I - GK_{\text{nom}})^{-1}P_{21} \\ T_2 &= -P_{12}(I - K_{\text{nom}}G)^{-1} \\ T_3 &= (I - GK_{\text{nom}})^{-1}P_{21} \end{aligned} \tag{3}$$

Proof. The proof is omitted due to space constraints. ■

This theorem tells us that if the plant is strongly stabilizable, that is, if it can be stabilized by a stable controller, then given such a controller, we can parameterize the set of all stabilizing controllers. See [18] for a discussion of this, [1] for an extension to nonlinear control, and [9] for an extension to decentralized control with sparsity constraints. The parameterization above is very useful, since in the absence of the constraint $K \in S$, problem (1) can be reformulated as

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2QT_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \end{aligned} \tag{4}$$

The closed-loop map is now affine in Q, and its norm is therefore a convex function of Q. This problem is readily solvable by, for example, the techniques in [3]. After solving this problem to find Q, one may then construct the optimal K for problem (1) via $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$.

Parameterization of all stabilizing controllers for decentralized control. We now wish to extend the above result to parameterize all stabilizing controllers $K \in \mathcal{R}_p$ that also satisfy the information constraint $K \in S$. Applying the above change of variables to problem (1), we arrive at the following optimization problem.

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2QT_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \in S \end{aligned} \tag{5}$$

However, the set

$$\Big\{ Q \in \mathcal{RH}_\infty \;\Big|\; K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \in S \Big\}$$

is not convex in general, and hence this problem is not easily solved. In this paper, we develop general conditions under which this set is convex.

2.3 Quadratic Invariance

There is no known tractable solution to the general problem (1) when S is an arbitrary subspace. However, the recent results of [8, 9] provide conditions under which the problem may be solved. These results say that, if the information constraint S is quadratically invariant, then problem (1) may be solved via convex optimization. We now state formally this property.

Definition 2. *The set S is called **quadratically invariant** under G if*

$$KGK \in S \qquad \textit{for all } K \in S$$

The following lemmas are proven in [8].

Lemma 3. *Suppose $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, and $S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$ is a subspace. If S is quadratically invariant under G, then*

$$K(GK)^n \in S \qquad \textit{for all } K \in S,\ n \in \mathbb{Z}_+$$

Lemma 4. *Suppose $\mathcal{U}, \mathcal{Y}$ are Banach spaces, $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y}), S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$, and S is not quadratically invariant under G. Then there exists $K \in S$ such that $(I - GK)$ is invertible and $K(I - GK)^{-1} \notin S$.*

The following is the main result of [8]. It states that given G, if we have any constraint set S which is quadratically invariant, then subject to technical conditions, the information constraints on K are equivalent to *affine constraints* on the map $h(K)$.

Theorem 5. *Suppose $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, and $S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$ is a closed subspace. Further suppose $N \cap S = M \cap S$. Then*

$$\begin{aligned} &S \textit{ is quadratically invariant under } G \\ &\qquad\qquad \iff h(S \cap M) = S \cap M \end{aligned}$$

2.4 Connectedness of the Resolvent Set

The technical conditions of Theorem 5 are automatically satisfied when the Banach spaces $\mathcal{U}$ and $\mathcal{Y}$ are finite dimensional, hence this result is directly applicable to controller synthesis subject to sparsity constraints [9]. However, these assumptions prevent immediate application to systems with delays. Further, the following example shows that these technical conditions are necessary in general. Let

$$\ell_2 = \left\{ (\ldots, x_{-1}, x_0, x_1, \ldots) \;;\; x_i \in \mathbb{R},\ \sum_{i=-\infty}^{\infty} x_i^2 < \infty \right\}.$$

Define $\ell_2^+ = \{x \in \ell_2 \;;\; x_i = 0 \text{ for all } i < 0\}$. and define the delay operator $D : \ell_2 \to \ell_2$ as $D(x)_i = x_{i-1}$. Let $\mathcal{Y} = \mathcal{U} = \ell_2$, let the plant be the identity $G = I$, and let S be the subspace of causal controllers

$$S = \big\{ K \in \mathcal{L}(\ell_2) \;;\; K(y) \in \ell_2^+ \text{ for all } y \in \ell_2^+ \big\}$$

such that S is clearly quadratically invariant under G. Now consider $K = 2D \in S$; we have

$$\begin{aligned} (I - GK)^{-1} &= -\frac{1}{2}D^{-1}\left(I - \frac{1}{2}D^{-1}\right)^{-1} \\ &= -\sum_{k=1}^{\infty} \frac{1}{2^k} D^{-k} \end{aligned}$$

and so $K \in M$. Also note that

$$\rho(GK) = \{\lambda \in \mathbb{C}\ ;\ |\lambda| \neq 2\}$$

and hence $\rho_{uc}(GK) = \{\lambda \in \mathbb{C}\ ;\ |\lambda| > 2\}$, which implies that $K \notin N$. Finally,

$$K(I - GK)^{-1} = -\sum_{k=0}^{\infty} \frac{1}{2^k} D^{-k} \notin S$$

So we have $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y}), S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$ is a closed subspace, and S is quadratically invariant under G, but $N \cap S \neq M \cap S$. We have then found a $K \in S \cap M$ such that $h(K) \notin S$, and so $h(S \cap M) \neq S \cap M$.

This elucidates the fact that the above technical conditions cannot be completely eradicated, and motivates us to find a different framework under which a similar result can be achieved without them. In the remainder of the paper, we achieve this by focusing on causal operators.

3 Invariance Under Feedback

3.1 Causal Operators

We would like to develop more general conditions under which the closed-loop system $K(I - GK)^{-1}$ lies in the information subspace S. In this section, we show that by focusing on causal operators, we can both extend our main results to unbounded operators and eliminate technical conditions from our assumptions.

3.1.1 Convergence of Neumann Series

To do this, we first analyze convergence of the *Neumann series*

$$(I - W)^{-1} = \sum_{n=0}^{\infty} W^n$$

when W is a general causal linear operator on extended spaces. In particular, we need to define much more general conditions for convergence of the Neumann series than the well-known small gain theorem. Note that while most of the results in paper have analogs in both continuous-time and discrete-time, the proofs in these cases are different. We first analyze the continuous-time case, and begin by providing a preliminary lemma relating the convergence of impulse responses with the convergence of their associated operators.

Lemma 6. *Suppose $W_n \in \mathcal{L}(L_{2e}^m)$ is causal and time-invariant for all $n \in \mathbb{Z}_+$, $w^{(n)} \in L_{\infty e}$ is the impulse response of W_n, $a \in L_{\infty e} \subseteq L_{1e}$ and $(w^{(n)})_T$ converges uniformly to a_T for all $T \in \mathbb{R}_+$. Then W_n converges to $A \in \mathcal{L}(L_{2e}^m)$, where A is given by $Au = a * u$.*

Proof. Given $u \in L_{2e}^m$ and $T \in \mathbb{R}_+$,

$$(a * u)_T = (a_T * u_T)_T$$

since $a(t) = 0$ and $u(t) = 0$ for $t < 0$. Hence $(a * u)_T \in L_2$, since $a_T \in L_1$ and $u_T \in L_2$, by Theorem 65 of [12]. Therefore, we can define $A \in \mathcal{L}(L_{2e}^m)$ by $Au = a * u$.

For any $n \in \mathbb{Z}_+$ and any $T \in \mathbb{R}_+$,

$$\begin{aligned} \|A - W_n\|_T^2 &= \sup_{u \in L_2, \|u\|_2 = 1} \|P_T A u - P_T W_n u\|_2^2 \\ &\leq \sup_{u \in L_2, \|u\|_2 = 1} \left\| \left(a_T - (w^{(n)})_T\right) * u \right\|_2^2 \\ &\leq \sup_{u \in L_2, \|u\|_2 = 1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left\| \left(a_T - (w^{(n)})_T\right)_{ij} \right\|_1^2 \|u_j\|_2^2 \end{aligned}$$

and hence

$$\|A - W_n\|_T^2 \leq \sum_{i=1}^{m} \sum_{j=1}^{m} \left\| \left(a_T - (w^{(n)})_T\right)_{ij} \right\|_1^2$$

Since the sum converges uniformly to a_T, for any $\epsilon > 0$ we can choose N such that for all $n \geq N$ and for all $i, j = 1, \ldots, m$, $\left|(a_{ij})_T(t) - \sum_{k=1}^{n} (w_{ij}^{(k)})_T(t)\right| < \frac{\epsilon}{mT}$ for all $t \in [0, T]$ and thus $\|A - W_n\|_T < \epsilon$. So W_n converges to A in $\mathcal{L}(L_{2e}^m)$. ■

We can now prove convergence of the Neumann series under the given conditions by showing the convergence of impulse responses. The method for showing this is similar to that used for spatio-temporal systems in the appendix of [2].

Theorem 7. *Suppose $W \in \mathcal{L}(L_{2e}^m)$ is causal and time-invariant with impulse response matrix w such that $w \in L_{\infty e}$. Then $\sum_{n=0}^{\infty} W^n$ converges to an element $B \in \mathcal{L}(L_{2e}^m)$ such that $B = (I - W)^{-1}$.*

Proof. Let $q(T) = \sup_{t \in [0,T]} \|w(t)\| < \infty$ for all $T \in \mathbb{R}_+$, and let $w^{(n)}$ be the impulse response matrix of W^n. First we claim that $\|w^{(n)}(T)\| \leq \frac{T^{n-1}}{(n-1)!}\, q(T)^n$ for all integers $n \geq 1$. This is true immediately for $n = 1$. For the inductive step,

$$\begin{aligned} \left\|w^{(n+1)}(T)\right\| &= \left\| \int_{t=0}^{T} w(T - t) w^{(n)}(t) dt \right\| \\ &\leq \int_{t=0}^{T} \|w(T - t)\| \cdot \|w^{(n)}(t)\| dt \\ &\leq q(T) \int_{t=0}^{T} \|w^{(n)}(t)\| dt \\ &\leq q(T) \int_{t=0}^{T} \frac{t^{n-1}}{(n-1)!} q(t)^n dt \\ &\leq \frac{T^n}{n!}\, q(T)^{n+1} \end{aligned}$$

Then $|w_{ij}^{(n)}(t)| \leq \frac{T^{n-1}}{(n-1)!}\, q(T)^n$ for all $t \in [0,T]$,for all $n \geq 1$, and for all $i,j = 1,\ldots,m$. $\sum_{n=1}^{\infty} \frac{T^{n-1}}{(n-1)!}\, q(T)^n$ converges to $q(T)e^{Tq(T)}$, so by the Weierstrass M-test, $\sum_{n=1}^{\infty}\big(w_{ij}^{(n)}\big)_T$ converges uniformly and absolutely for all $i,j = 1,\ldots,m$.

Let $a = \sum_{n=1}^{\infty} w^{(n)}$. Then $a_{ij} \in L_{\infty e} \subseteq L_{1e}$ for all $i,j = 1,\ldots,m$, and we can define $A, B \in \mathcal{L}(L_{2e}^m)$ by $Au = a * u$ and $B = I + A$.

Then by Lemma 6, $\sum_{k=1}^{n} W^k$ converges to A in $\mathcal{L}(L_{2e}^m)$, and thus $\sum_{k=0}^{n} W^k$ converges to B in $\mathcal{L}(L_{2e}^m)$.

Lastly,

$$B(I-W) = (I-W)B = \sum_{n=0}^{\infty} W^n - \sum_{n=1}^{\infty} W^n = I$$

■

A simple example of the utility of this result is as follows. Consider W represented by the transfer function $\frac{2}{s+1}$. Then $I - W = \frac{s-1}{s+1}$ is not invertible in $\mathcal{L}(L_2)$. However using the above theorem, the inverse in $\mathcal{L}(L_{2e})$ is given by $\sum_{n=0}^{\infty}(\frac{2}{s+1})^n = \frac{s+1}{s-1}$.

We now move on to analyze the discrete-time case. Let $\mathrm{rad}(\cdot)$ denote spectral radius.

Theorem 8. *Suppose $W \in \mathcal{L}(\ell_e^m)$ is causal and time-invariant with impulse response matrix w such that $w \in \ell_e$ and $\mathrm{rad}(w(0)) < 1$. Then $\sum_{n=0}^{\infty} W^n$ converges to an element $B \in \mathcal{L}(\ell_e^m)$ such that $B = (I-W)^{-1}$.*

Proof. We may represent $P_T W|_T$ with the block lower triangular Toeplitz matrix

$$W_T = \begin{bmatrix} w(0) & & \\ w(1) & \ddots & \\ \vdots & & \\ w(T) & \cdots & w(0) \end{bmatrix}$$

Since $w \in \ell_e$, $W_T \in \mathbb{R}^{mT \times mT}$. Then, $\mathrm{rad}(W_T) = \mathrm{rad}(w(0)) < 1$, which implies that $\sum_{n=0}^{\infty}(W_T)^n$ converges in $\mathbb{R}^{mT\times mT}$. Thus we can define $B \in \mathcal{L}(\ell_e^m)$ by $(Bu)_T = (\sum_{n=0}^{\infty}(W_T)^n)u_T$ for any $u \in \ell_e^m$ and any $T \in \mathbb{Z}_+$. It is then immediate that $\left\| B - \sum_{n=0}^{\infty} W^n \right\|_T \to 0$ for all T, and thus $\sum_{n=0}^{\infty} W^n$ converges to B in $\mathcal{L}(\ell_e^m)$.

Lastly,

$$B(I-W) = (I-W)B = \sum_{n=0}^{\infty} W^n - \sum_{n=1}^{\infty} W^n = I$$

■

Note that while the conditions of Theorem 8 are necessary for convergence as well as sufficient, the conditions of Theorem 7 are not.

In particular, the above results imply the following corollaries, which show convergence of the Neumann series for strictly proper systems, possibly with delay.

Corollary 9. *Suppose $W \in \mathcal{L}(L_{2e}^m)$ is given by $W_{ij} = D_{\tau_{ij}} G_{ij}$ where $\tau_{ij} \geq 0$ and $G_{ij} \in \mathcal{R}_{sp}$. Then $\sum_{n=0}^{\infty} W^n$ converges to an element $B \in \mathcal{L}(L_{2e}^m)$ such that $B = (I-W)^{-1}$.*

Corollary 10. *Suppose $W \in \mathcal{L}(\ell_e^m)$ is given by $W_{ij} \in \mathcal{R}_{sp}$. Then $\sum_{n=0}^{\infty} W^n$ converges to an element $B \in \mathcal{L}(\ell_e^m)$ such that $B = (I-W)^{-1}$.*

4 Main Results

This subsection contains the main technical results of this paper. In particular, we show that for a broad class of systems, quadratic invariance allows convex synthesis for decentralized control. Specifically, we do not require a constraint on the resolvent set of a bounded operator, nor a structure constraint on the information subspace S.

We first state a lemma which will help with the converse of our main result.

Lemma 11. *Suppose $S \subseteq \mathcal{L}(L_{2e}^m, L_{2e}^n)$ or $S \subseteq \mathcal{L}(\ell_e^m, \ell_e^n)$, and $C \notin S$. Then there exists T such that $C_T \notin S_T$.*

Proof. Suppose not. Then for every positive T, $C_T \in S_T$. Thus for every T, there exists $K \in S$ such that $P_T C|_T = P_T K|_T$, or $\|C - K\|_T = 0$. Since $\|A\|_T = 0$ only if $\|A\|_\tau = 0$ for all $\tau \leq T$, it follows that there exists $K \in S$ such that $\|C-K\|_T = 0$ for all T. But then $C - K = 0$, and so $C \in S$ and we have a contradiction.
■

Definition 12. *We say that $S \subseteq \mathcal{L}(L_{2e}^{n_u}, L_{2e}^{n_y})$ is **inert** if for all $K \in S$, $(gk)_{ij} \in L_{\infty e}$ for all $i,j = 1,\ldots,m$ where (gk) is the impulse response matrix of GK. We overload our notation and also call $S \subseteq \mathcal{L}(\ell_e^{n_u}, \ell_e^{n_y})$ an* inert *subspace if for all $K \in S$, $(gk)_{ij} \in \ell_e$ for all $i,j = 1,\ldots,m$ and $\mathrm{rad}((gk)(0)) < 1$ where (gk) is the discrete impulse response matrix of GK.*

Theorem 13. *Suppose $G \in \mathcal{L}(L_{2e}^{n_u}, L_{2e}^{n_y})$ or $G \in \mathcal{L}(\ell_e^{n_u}, \ell_e^{n_y})$, and S is an inert closed subspace. Then*

$$S \textit{ is quadratically invariant under } G \iff h_G(S) = S$$

Proof. ($\Longrightarrow$) Suppose $K \in S$. We first show that $h_G(K) \in S$.

$$K(I-GK)^{-1} = K\sum_{n=0}^{\infty}(GK)^n = \sum_{n=0}^{\infty} K(GK)^n$$

where the first equality follows from Theorems 7 and 8 and the second follows from the continuity of K.

By Lemma 3 we have $K(GK)^n \in S$ for all $n \in \mathbb{Z}_+$, and hence $K(I-GK)^{-1} \in S$ since S is a closed subspace.

So $K \in S \implies h_G(K) \in S$. Thus $h_G(S) \subseteq S$, and since h_G is involutive it follows that $h_G(S) = S$, which was the desired result.

($\Longleftarrow$) We now turn to the converse of this result. Suppose that S is not quadratically invariant under G. Then there exists $K \in S$ such that $KGK \notin S$, and thus by Lemma 11, there exists a finite T such that $P_T KGK|_T \notin S_T$. Since K and G are causal, we then have

$$K_T G_T K_T \notin S_T \qquad \text{where} \qquad K_T = P_T K P_T \in S_T \quad \text{and} \quad G_T = P_T G P_T$$

and thus S_T is not quadratically invariant under G_T. Then by Lemma 4 there exists $\tilde{K} \in S_T$ such that

$$\tilde{K}(I - G_T\tilde{K})^{-1} = \sum_{n=0}^{\infty} \tilde{K}(G_T\tilde{K})^n \notin S_T$$

By definition of S_T, there exists $K_0 \in S$ such that $\tilde{K} = P_T K_0|_T$. Then by causality of K_0 and G,

$$P_T\left(\sum_{n=0}^{\infty} K_0(GK_0)^n\right)\Bigg|_T \notin S_T$$

and thus $h_G(K_0) = -\sum_{n=0}^{\infty} K_0(GK_0)^n \notin S$. ■

The following theorem states that if the constraint set is quadratically invariant under the plant, and Q is defined as above, then the information constraints on K are equivalent to *affine constraints* on Q. Here K_{nom} is a stable stabilizing controller that satisfies the information constraints, i.e., $K_{\text{nom}} \in S$.

Theorem 14. *Suppose $G \in \mathcal{R}_{sp}$ and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$. If S is quadratically invariant under G then*

$$C_{\text{stab}} \cap S = \Big\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \;\big|\; Q \in \mathcal{RH}_\infty \cap S \Big\}$$

Proof. The full proof is omitted due to space constraints. Given $G \in \mathcal{R}_{sp}$, $S \subseteq \mathcal{R}_p$ is inert, and then once given $h_G(S) = S$ from Theorem 13, the rest of the proof follows that of Theorem 8 in [9] once given $h_G(S) = S$ from Theorem 6. ■

Equivalent convex problem. When the constraint set is quadratically invariant under the plant, we now have the following equivalent problem. Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ and $S \subseteq \mathcal{R}_p^{n_u \times n_y}$ is a closed subspace. Then K is optimal for problem (1) if and only if $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$ and Q is optimal for

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & Q \in S \end{aligned} \tag{6}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$ are given by equations (3). This problem may be solved via convex programming.

5 Distributed Control With Delays

We now consider the distributed control problem discussed in Section 2.1. Suppose there are n subsystems with transmission delay $t \geq 0$, propagation delay $p \geq 0$ and computational delay $c \geq 0$. When expressed in linear-fractional form, we define the allowable set of controllers is as follows. Let $K \in S$ if and only if

$$K = \begin{bmatrix} D_c H_{11} & D_{t+c} H_{12} & \dots & D_{(n-1)t+c} H_{1n} \\ D_{t+c} H_{21} & D_c H_{22} & \dots & D_{(n-2)t+c} H_{2n} \\ \vdots & & & \vdots \\ D_{(n-1)t+c} H_{n1} & \dots & & D_c H_{nn} \end{bmatrix}$$

for some $H_{ij} \in \mathcal{R}_p$ of appropriate spatial dimensions. The corresponding system G is given by

$$G = \begin{bmatrix} A_{11} & D_p A_{12} & \dots & D_{(n-1)p} A_{1n} \\ D_p A_{21} & A_{22} & \dots & D_{(n-2)p} A_{2n} \\ \vdots & & & \vdots \\ D_{(n-1)p} A_{n1} & \dots & & A_{nn} \end{bmatrix}$$

for some $A_{ij} \in \mathcal{R}_{sp}$.

We define Delay$(\cdot)$ to give the delay associated with a causal operator

$$\text{Delay}(W) = \arg\inf_{\tau > 0} w(\tau) \neq 0 \qquad \text{where } w \text{ is the impulse response of } W$$

Theorem 15. *Suppose that G and S are defined as above, and $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$. Then if*

$$t \leq p + \frac{c}{(n-1)}$$

we have

$$C_{\text{stab}} \cap S = \Big\{ K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big) \;\big|\; Q \in \mathcal{RH}_\infty \cap S \Big\}.$$

Proof. Given $K \in S$,

$$KGK \in S \iff \text{Delay}\big((KGK)_{kl}\big) \geq c + t|k-l| \text{ for all } k, l$$

We now seek conditions which cause this to hold.

$$(KGK)_{kl} = \sum_i \sum_j K_{kl} G_{ij} K_{jl}$$

and so, assuming w.l.o.g. that $k \leq l$,

$$\begin{aligned} &\text{Delay}\big((KGK)_{kl}\big) \\ &\qquad \geq \min_{i,j} \{\text{Delay}(K_{kl}) + \text{Delay}(G_{ij}) + \text{Delay}(K_{jl})\} \\ &\qquad \geq \min_{i,j} \{2c + t(|k-i| + |j-l|) + p|i-j|\} \\ &\qquad = \min_{k \leq i,j \leq l} \{2c + t(|k-i| + |j-l|) + p|i-j|\} \\ &\qquad = \min_{k \leq i,j \leq l} \{2c + t(|k-l| - |i-j|) + p|i-j|\} \\ &\qquad = 2c + t|k-l| + \min_{k \leq i,j \leq l} \{(p-t)|i-j|\} \\ &\qquad = 2c + \min\{t, p\}|k-l| \end{aligned}$$

So the condition for quadratic invariance is

$$2c + \min\{t, p\}|k - l| \geq c + t|k - l| \text{ for all } k, l.$$

This is equivalent to $c - (t - \min\{t, p\})(n - 1) \geq 0$, which is equivalent to $t \leq p + c/(n - 1)$. So when this inequality holds, S is quadratically invariant under G, and the desired result follows from Theorem 14. ■

Thus we see that finding the minimum-norm controller may be reduced to the convex optimization problem (6) when the controllers can transmit information faster than the dynamics propagate. We also see that the presence of computational delay causes this condition to be relaxed. In the case where the penalty of interest is the $\mathcal{H}_2$-norm, the explicit computational methods delineated in [10] for synthesizing the optimal controller subject to quadratically invariant sparsity constraints are easily extended to delay constraints.

6 Conclusions

We have developed a new framework for the analysis and synthesis of minimum-norm decentralized control problems, with causality as the only main assumption. We showed that in this new framework, synthesis of the minimum-norm controller subject to *quadratically invariant* information constraints may be reduced to a convex optimization problem. This result holds for stable and unstable systems, for continuous and discrete-time, and is free from the strictures of extra conditions which existed when analyzed in more conventional frameworks. This enables a complete proof of convexity for an important pragmatic example, as we showed that optimal controllers for distributed systems with delays may be synthesized in this manner when the communication delay is less than the propagation delay.

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