A Simple Condition for the Convexity of Optimal Control over Networks with Delays

Michael Rotkowitz
Aeronautics and Astronautics
Stanford University
Stanford CA 94305-4035 USA
Email: rotkowitz@stanford.edu

Randy Cogill
Electrical Engineering
Stanford University
Stanford CA 94305-9505 USA
Email: rcogill@stanford.edu

Sanjay Lall
Aeronautics and Astronautics
Stanford University
Stanford CA 94305-4035 USA
Email: lall@stanford.edu

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Abstract— **We consider the problem of multiple subsystems, each with its own controller, such that the dynamics of each subsystem may affect those of other subsystems with some propagation delays, and the controllers may communicate with each other with some transmission delays. We wish to synthesize controllers to minimize a closed-loop norm for the entire system. We show that if the transmission delays satisfy the triangle inequality, then the simple condition that the transmission delay between any two subsystems is less than the propagation delay between those subsystems allows for the optimal control problem to be recast as a convex optimization problem.**

I. Introduction

We consider the problem of multiple subsystems, each with its own controller, such that the dynamics of each subsystem may effect those of other subsystems with some propagation delay, and the controllers may communicate with each other with some transmission delays. We seek to synthesize linear controllers to minimize a closed-loop norm for the entire interconnected system. This is an optimal decentralized control problem which is difficult in general, and there is no known tractable solution for arbitrary propagation and transmission delays. This paper states simple conditions on the delays such that this optimal control problem may be cast as a convex optimization problem.

It has been shown for general decentralized control that a property called quadratic invariance allows the optimal control problem to be recast as a convex optimization problem [1]. We thus achieve our characterization of delays which allow for convex synthesis by testing for quadratic invariance.

We find that if the transmission delays satisfy the triangle inequality, and if the propagation delay between any pair of subsystems is at least as large as the transmission delay between those subsystems, then the problem is quadratically invariant. In other words, if data can be transmitted faster than dynamics propagate along any link, then optimal controller synthesis may be formulated as a convex optimization problem.

It is important to note the extreme generality of this framework and of this result. It holds for discrete-time systems and continuous-time systems. It holds for any norm that we wish to minimize. It does not assume that the dynamics of any subsystem are the same as those of any other, and they may all be completely different types of objects. Most importantly, the delay between any two subsystems is not assumed to have any relationship whatsoever to other delays in the system. They may be assigned independently for each link. Only in the examples do we assume otherwise.

a) Prior Work: A vast amount of prior work on optimal control over networks assumes that the actions of any subsystem have no effect on the dynamics of other subsystems. For a few other specific structures, tractable methods have been found. One of the first problems of this nature to be studied was the one-step delayed information sharing problem. This problem assumes that each subsystem has a controller that can see its own output immediately, and can see outputs from all other subsystems after a delay of one time step. This problem has long been known to admit tractable solutions [2], and has also been studied more recently in an LFT framework [3]. An interesting class of spatio-temporal systems which allow for convex synthesis of optimal controllers was identified in [4], and named funnel causal systems. One of the tractable structures discussed in [5] involved evenly spaced subsystems which can pass measurements on at the same speed that the dynamics propagate, and [1] included a similar class of evenly spaced systems where the bound was found such that if the communication speed exceeded that bound the problem was amenable to convex synthesis.

These results are all unified and generalized by the simple conditions found in this paper.

b) Outline: In Section II, we state some preliminaries and notation, define the propagation and transmission delays, explain why we may assume that the transmission delays satisfy the triangle inequality, formulate the problem we wish to solve, and give an overview of results on quadratic invariance, in particular, that it allows convex synthesis of optimal linear decentralized controllers.

Section III contains the main result of the paper, where we prove that if this triangle inequality is satisfied, and if the propagation delay associated with any pair of subsystems is at least as large as the associated transmission delay, then the information constraint is quadratically invariant, and

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thus, optimal control may be cast as a convex optimization problem.

In Section III-A we break these total transmission delays out into a pure transmission delay, representing the time it takes to communicate the information from one subsystem to another, and a computational delay, representing the time it takes to process the information before it is used by the controller. We find, somewhat surprisingly, that transmitting faster than the propagation of dynamics still guarantees convexity, and in fact, that the computational delay causes the condition to be relaxed.

In Section III-B, we discuss how sparsity constraints may be considered a special case of the framework analyzed in this paper, namely, by viewing them as very large delays. We then show how sparsity and delay constraints can be combined to handle the very general, realistic case of a network where some nodes are connected with delays and others are not connected at all.

We then consider a few examples in Section IV. First is an example corresponding to a very general problem of the control of vehicles in formation. The vehicles may have arbitrary positions, their dynamics propagate at a constant speed, and they communicate their measurements at a constant speed. The optimal control problem is amenable to convex synthesis as long as the communication speed exceeds the propagation speed. Even though this itself is a broad generalization of previously identified tractable classes, it follows almost immediately from the results of this paper. Conditions are then derived for convexity of optimal control over a lattice, for two different types of assumptions on the propagation of dynamics.

We make some concluding remarks in Section V.

II. PRELIMINARIES

We define $\mathrm{Delay}(\cdot)$ to give the delay associated with a time-invariant causal operator

$$\mathrm{Delay}(W) \;=\; \inf\{\tau \geq 0 \mid w(\tau) \neq 0\}$$
$$\text{where } w \text{ is the impulse response of } W$$

Note that we then have the following inequalities for the delays of a composition or an addition of operators:

$$\mathrm{Delay}(AB) \;\geq\; \mathrm{Delay}(A) + \mathrm{Delay}(B)$$
$$\mathrm{Delay}(A+B) \;\geq\; \min\{\mathrm{Delay}(A), \mathrm{Delay}(B)\}$$

We suppose that we have a generalized plant P partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

We define the ***closed-loop map*** by

$$f(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

The map $f(P,K)$ is also called the (lower) ***linear fractional transformation*** (LFT) of P and K. In the remainder of the paper, we abbreviate our notation and define $G = P_{22}$, since we will refer to that block frequently. This interconnection is shown in Figure 1.

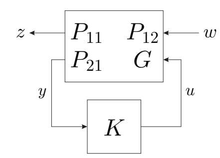

Fig. 1. Linear fractional interconnection of P and K

We suppose that there are n subsystems, each with its own controller, and thus partition the sensor measurements and control actions as

$$y = \begin{bmatrix} y_1^T & \dots & y_n^T \end{bmatrix}^T \qquad u = \begin{bmatrix} u_1^T & \dots & u_n^T \end{bmatrix}^T$$

and then further partition G and K as

$$G = \begin{bmatrix} G_{11} & \dots & G_{1n} \\ \vdots & & \vdots \\ G_{n1} & \dots & G_{nn} \end{bmatrix} \qquad K = \begin{bmatrix} K_{11} & \dots & K_{1n} \\ \vdots & & \vdots \\ K_{n1} & \dots & K_{nn} \end{bmatrix}$$

A. Propagation Delays

For any pair of subsystems i and j we define the propagation delay p_{ij} as the amount of time before a controller action at subsystem j can affect an output at subsystem i as such

$$p_{ij} \;=\; \mathrm{Delay}(G_{ij}) \qquad \text{for all } i,j \in 1,\dots,n$$

B. Transmission Delays

For any pair of subsystems k and l we define the (total) transmission delay t_{kl} as the minimum amount of time before the controller of subsystem k may use outputs from subsystem l. Given these constraints, we can define the overall subspace of admissible controllers S such that $K \in S$ if and only if

$$\mathrm{Delay}(K_{kl}) \;\geq\; t_{kl} \qquad \text{for all } k,l \in 1,\dots,n$$

In Section III-A we will break these total transmission delays out into a pure transmission delay, representing the time it takes to communicate the information from one subsystem to another, and a computational delay, representing the time it takes to process the information before it is used by the controller.

c) Triangle inequality: For the main result of this paper, we will assume that the triangle inequality holds amongst the transmission delays, that is,

$$t_{ki} + t_{ij} \;\geq\; t_{kj} \qquad \text{for all } k,i,j$$

This is typically a very reasonable assumption for the following reasons. t_{kj} is defined as the minimum amount of time before controller k can use outputs from subsystem j. So if there existed an i such that the inequality above failed, that would mean that controller k could receive that information more quickly if it came indirectly via controller i. We would thus reroute this information to go through i, t_{kj} would be reset to $t_{ki} + t_{ij}$, and the inequality would hold.

To put it another way, we could think of each subsystem as a node on a directed graph, with the initial distance from any node j to any node k as t_{kj}, the time it takes before controller k can directly use outputs from subsystem j. We then want to find the shortest overall time for any controller k to use outputs from any subsystem j, that is, the shortest path from node j to node k. So to find our final t_{kj}'s, we run Bellman-Ford or another shortest path algorithm on our initial graph [6], and the resulting delays are thus guaranteed to satisfy the triangle inequality.

C. Problem Formulation

Given a generalized plant P and transmission delays t_{kl} for each pair of subsystems, we define S as above, and we would then like to solve the following problem:

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned} \tag{1}$$

Here $\|\cdot\|$ is any norm on the closed-loop map chosen to encapsulate the control performance objectives. The delays associated with dynamics propagating from one subsystem to another are embedded in P. The subspace of admissible controllers, S, has been defined to encapsulate the constraints on how quickly information may be passed from one subsystem to another. We call the subspace S the ***information constraint***.

Many decentralized control problems may be expressed in the form of problem (1). In this paper, we focus on the case where S is defined by delay constraints as discussed above.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved with many standard techniques. Note that the cost function $\|f(P,K)\|$ is in general a non-convex function of K. No computationally tractable approach is known for solving this problem for arbitrary P and S.

D. Quadratic Invariance

In this subsection we define quadratic invariance, and give a brief overview of results regarding this condition, in particular, that it allows convex synthesis of optimal linear decentralized controllers.

Definition 1: The set S is called ***quadratically invariant*** under G if

$$KGK \in S \qquad \text{for all } K \in S$$

Note that, given G, we can define a quadratic map by $\Psi(K) = KGK$. Then a set S is quadratically invariant if and only if S is an invariant set of Ψ; that is $\Psi(S) \subseteq S$.

It was shown in [1] that if S is a closed subspace and S is quadratically invariant under G, then with a change of variables, problem (1) is equivalent to the following optimization problem

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & Q \in S \end{aligned} \tag{2}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$.

This is a convex optimization problem. We may solve it to find the optimal Q, and then recover the optimal K for our original problem.

If the norm of interest is the $\mathcal{H}_2$-norm, it was shown in [1] that the problem can be further reduced to an unconstrained optimal control problem and then solved with standard software.

We have assumed for this overview that these operators are all real-rational proper and thus acting on L_{2e} or ℓ_e. Similar results have been achieved [7] for other function spaces as well, also showing that quadratic invariance allows optimal linear decentralized control problems to be recast as convex optimization problems.

The main focus of this paper is thus characterizing delays for which the information constraint S is quadratically invariant under the plant G.

III. Conditions for Convexity

We first provide a necessary and sufficient condition for quadratic invariance in terms of these delays, which is derived fairly directly from our definitions.

Theorem 2: Suppose that G and S are defined as above. S is quadratically invariant under G if and only if

$$t_{ki} + p_{ij} + t_{jl} \;\geq\; t_{kl} \qquad \text{for all } i,j,k,l \tag{3}$$

Proof: Given $K \in S$,

$$KGK \in S \iff \text{Delay}\big((KGK)_{kl}\big) \geq t_{kl} \text{ for all } k,l$$

We now seek conditions which cause this to hold.

$$(KGK)_{kl} = \sum_i \sum_j K_{ki} G_{ij} K_{jl}$$

and so for any k and l,

$$\begin{aligned} &\text{Delay}\big((KGK)_{kl}\big) \\ &\quad \geq \min_{i,j} \{\text{Delay}(K_{ki} G_{ij} K_{jl})\} \\ &\quad \geq \min_{i,j} \{\text{Delay}(K_{ki}) + \text{Delay}(G_{ij}) + \text{Delay}(K_{jl})\} \\ &\quad \geq \min_{i,j} \{t_{ki} + p_{ij} + t_{jl}\} \end{aligned}$$

Thus S is quadratically invariant under G if

$$\min_{i,j} \{t_{ki} + p_{ij} + t_{jl}\} \;\geq\; t_{kl} \qquad \text{for all } k,l$$

which is equivalent to

$$t_{ki} + p_{ij} + t_{jl} \;\geq\; t_{kl} \qquad \text{for all } i,j,k,l$$

Now suppose that Condition (3) fails. Then there exists i,j,k,l such that

$$t_{ki} + p_{ij} + t_{jl} \;<\; t_{kl}$$

Consider K such that

$$K_{ab} = 0 \text{ if } (a,b) \notin \{(k,i),(j,l)\}$$

Then

$$(KGK)_{kl} \;=\; \sum_r \sum_s K_{kr} G_{rs} K_{sl} \;=\; K_{ki} G_{ij} K_{jl}$$

Since $\mathrm{Delay}(G_{ij}) = p_{ij}$, we can easily choose K_{ki} and K_{jl} such that $\mathrm{Delay}(K_{ki}) = t_{ki}$, $\mathrm{Delay}(K_{jl}) = t_{jl}$, and

$$\mathrm{Delay}\big((KGK)_{kl}\big) = t_{ki} + p_{ij} + t_{jl}$$

So $K \in S$ but $KGK \notin S$ and thus S is not quadratically invariant under G. ■

Main Result. The following is the main result of this paper. It states that if the transmission delays satisfy the triangle inequality, and if the propagation delay between any pair of subsystems is at least as large as the transmission delay between those subsystems, then the information constraint is quadratically invariant. In other words, if along any link, data can be transmitted faster than dynamics propagate, then optimal controller synthesis may be cast as a convex optimization problem.

Theorem 3: Suppose that G and S are defined as above, and that the transmission delays satisfy the triangle inequality. If

$$p_{ij} \;\geq\; t_{ij} \qquad \text{for all } i,j \tag{4}$$

then S is quadratically invariant under G.

Proof: Suppose Condition (4) holds. Then for all i,j,k,l we have

$$\begin{aligned} t_{ki} + p_{ij} + t_{jl} \;&\geq\; t_{ki} + t_{ij} + t_{jl} \\ &\geq\; t_{kl} \qquad \text{by the triangle inequality} \end{aligned}$$

and thus by Theorem 2, S is quadratically invariant under G. ■

Thus we have shown that the triangle inequality and Condition (4) are sufficient for quadratic invariance. The following remarks discuss assumptions under which they are necessary as well.

Remark 4: If we assume that $t_{ii} = 0$ for all i, that is, that there is no delay before a subsystem's controller may use its own outputs, then we consider Condition (3) with $k = i,\ l = j$ and see that Condition (4) is necessary for quadratic invariance.

Remark 5: If we assume that $p_{ii} = 0$ for all i, that is, that there is no delay associated with propagating from a subsystem to itself, then we consider Condition (3) with $i = j$ and see that the triangle inequality is necessary for quadratic invariance.

A. Computational Delays

In this section, we consider what happens when the controller of each subsystem has a computational delay c_i associated with it. The delay for controller i to use outputs from subsystem j, the total transmission delay, is then broken up into a pure transmission delay and this computational delay, as follows

$$t_{ij} = c_i + \tilde{t}_{ij}$$

If we were to assume that the triangle inequality held for the total transmission delays t_{ij} as before, then we would simply get the same results as in the previous section with the substitution above. In particular, we would find $p_{ij} \geq c_i + \tilde{t}_{ij}$ to be the condition for quadratic invariance. However, there are many cases where it makes sense to instead assume that the triangle inequality holds for the pure transmission delays $\tilde{t}_{ij}$, which is a stronger assumption. An example where such is clearly the case is provided in Section IV-A.

In this section we derive conditions for quadratic invariance when we can assume that the triangle inequality holds for the pure transmission delays $\tilde{t}_{ij}$, and get a surprising result.

As before, the propagation delays are defined as

$$p_{ij} \;=\; \mathrm{Delay}(G_{ij}) \qquad \text{for all } i,j$$

and S is now defined such that $K \in S$ if and only if

$$\mathrm{Delay}(K_{kl}) \;\geq\; c_k + \tilde{t}_{kl} \qquad \text{for all } k,l$$

Thus the necessary and sufficient condition for quadratic invariance from Theorem 2 becomes

$$c_k + \tilde{t}_{ki} + p_{ij} + c_j + \tilde{t}_{jl} \;\geq\; c_k + \tilde{t}_{kl} \qquad \text{for all } i,j,k,l$$

which reduces to

$$\tilde{t}_{ki} + p_{ij} + c_j + \tilde{t}_{jl} \;\geq\; \tilde{t}_{kl} \qquad \text{for all } i,j,k,l \tag{5}$$

The following theorem gives conditions under which the information constraint is quadratically invariant. It states that if the triangle inequality holds amongst the pure transmission delays, and if Condition (6) holds, then the information constraint is quadratically invariant. Surprisingly, we see that the computational delay now appears on the left side of the inequality. In other words, not only does transmitting data faster than dynamics propagate still allow for convex synthesis when we account for computational delay, but the condition is actually relaxed.

Theorem 6: Suppose that G and S are defined as above, and that the pure transmission delays satisfy the triangle inequality. If

$$p_{ij} + c_j \;\geq\; \tilde{t}_{ij} \qquad \text{for all } i,j \tag{6}$$

then S is quadratically invariant under G.

Proof: Suppose Condition (6) holds. Then for all i,j,k,l we have

$$\begin{aligned} \tilde{t}_{ki} + p_{ij} + c_j + \tilde{t}_{jl} \;&\geq\; \tilde{t}_{ki} + \tilde{t}_{ij} + \tilde{t}_{jl} \\ &\geq\; \tilde{t}_{kl} \qquad \text{by the triangle inequality} \end{aligned}$$

and thus Condition (5) holds and S is quadratically invariant under G. ■

Thus we have shown that the triangle inequality and Condition (6) are sufficient for quadratic invariance. The following remark discusses an assumption under which the condition is necessary as well.

Remark 7: If we assume that $\tilde{t}_{ii} = 0$ for all i, that is, that there is no additional delay before a subsystem's controller may use its own outputs, other than the computational delay, then we consider Condition (5) with $k = i,\ l = j$ and see that Condition (6) is necessary for quadratic invariance. Since the computational delay has been extracted, this is now a very reasonable assumption which is essentially true by definition.

B. Combining Sparsity and Delay Constraints

In this section, we discuss how sparsity constraints may be considered a special case of the framework analyzed in this paper. We then show how the two can be combined to handle the very general, realistic case of a network where some nodes are connected with delays as above and others are not connected at all. An explicit test for quadratic invariance in this case is provided.

The key observation is that a sparsity constraint may be considered an infinite delay. We thus define an extended notion of propagation and transmission delays, where they are assigned to be sufficiently large when they do not exist,and then the results from the rest of this paper may be applied to test for quadratic invariance and convexity.

1) Propagation Delays: We now consider a plant for which the controllers of certain subsystems may or may not have any effect on other subsystems, and when they do, there may be a propagation delay associated with that effect. First, define a binary matrix G^{bin} such that

$$G_{ij}^{\text{bin}} = \begin{cases} 0 & \text{if } G_{ij} = 0 \\ 1 & \text{otherwise} \end{cases}$$

In other words, G^{bin} defines the sparsity structure or interconnection structure of the plant, as $G_{ij}^{\text{bin}} = 0$ if subsystem i is not affected by inputs to subsystem j. We would then like to define the propagation delay p_{ij} to be extremely large if this is the case, as such

$$p_{ij} = \begin{cases} H & \text{if } G_{ij}^{\text{bin}} = 0 \\ \text{Delay}(G_{ij}) & \text{if } G_{ij}^{\text{bin}} = 1 \end{cases}$$

for some large H.

2) Transmission Delays: We similarly assign a binary matrix K^{bin} such that $K_{kl}^{\text{bin}} = 0$ if controller k may never use outputs from subsystem l. For any other pair of subsystems k and l we define the (total) transmission delay t_{kl} as in the rest of this paper; that is, as the minimum amount of time before the controller of subsystem k may use outputs from subsystem l. Given these constraints, we can define the overall subspace of admissible controllers S such that $K \in S$ if and only if

$$\begin{aligned} K_{kl} &= 0 && \text{for all } k,l && \text{such that } K_{kl}^{\text{bin}} = 0 \\ \text{Delay}(K_{kl}) &\geq t_{kl} && \text{for all } k,l && \text{such that } K_{kl}^{\text{bin}} = 1 \end{aligned}$$

We wish to assign a very large transmission delay to the former case, and so define

$$t_{kl} = H \qquad \text{for all } k,l \quad \text{such that } K_{kl}^{\text{bin}} = 0$$

for the same large H as above.

3) Condition for Convexity: Given these extended definitions of propagation delays and transmission delays for a combination of sparsity and delay constraints, we can now test for quadratic invariance using Theorem 2.

These definitions of extended delays along with our definition of the constraint set S allow us to use this and the rest of the results of this paper as long as H has been chosen large enough. Condition (3) is indeed necessary and sufficient for quadratic invariance as long as

$$H \;>\; 2\max\{t_{kl}\} + \max\{p_{ij}\}$$

where of course the first maximum is taken over all k,l such that $K_{kl}^{\text{bin}} = 1$ and the second is taken over all i,j such that $G_{ij}^{\text{bin}} = 1$. The bound on H arises because Condition (3) must fail if $K_{kl}^{\text{bin}} = 0$, but $K_{ki}^{\text{bin}} = G_{ij}^{\text{bin}} = K_{jl}^{\text{bin}} = 1$.

IV. Examples

We consider here some special cases of interest.

A. Vehicle Formation Example

We now consider an important special case, which corresponds to the problem of controlling multiple vehicles in a formation.

Suppose there are n subsystems (vehicles), with positions $x_1, \ldots, x_n \in \mathbb{R}^d$. Typically, we'll have $d = 3$, but these results hold for arbitrary $d \in \mathbb{Z}_+$.

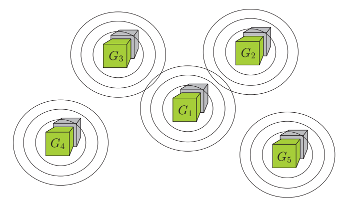

Fig. 2. Communication and propagation in all directions

Let R represent the maximum distance between any two subsystems

$$R = \max_{i,j} \|x_i - x_j\|$$

For most applications of interest the appropriate norm throughout this section would be the Euclidean norm, but these results hold for arbitrary norm on $\mathbb{R}^d$.

We suppose that dynamics of all vehicles propagate at a constant speed, determined by the medium, such that the propagation delays are proportional to the distance between vehicles, as illustrated in Figure 2.

Let γ_p be the amount of time it takes dynamics to propagate one unit of distance, i.e., the inverse of the speed of propagation. For example, when considering formations of aerial vehicles, γ_p would equal the inverse of the speed of sound.

The system G is then such that

$$\text{Delay}(G_{ij}) = \gamma_p \|x_i - x_j\| \qquad \text{for all } i,j$$

We similarly suppose that data can be transmitted at a constant speed, such that the transmission delays are proportional to the distances between vehicles, such as if each vehicle could broadcast its information to the others. This is also illustrated in Figure 2. We assume that the perturbations from our desired formation are small enough that, for the purposes of controller synthesis, we may consider these delays to be fixed.

Let γ_t be the amount of time it takes to transmit one unit of distance, i.e., the inverse of the speed of transmission.

Let C be the computational delay at each vehicle. The set of admissible controllers is then defined such that $K \in S$ if and only if

$$\mathrm{Delay}(K_{kl}) \;\geq\; C + \gamma_t \|x_k - x_l\| \qquad \text{for all } k, l$$

We can now apply Theorem 6 with

$$p_{ij} = \gamma_p \|x_i - x_j\|, \quad \tilde{t}_{ij} = \gamma_t \|x_i - x_j\|, \qquad \text{and } c_i = C \quad \text{for all } i, j$$

Clearly, $\tilde{t}_{ii} = 0$ for all i as in Remark 7, so the conditions of Theorem 6 are both necessary and sufficient for quadratic invariance.

Theorem 8: Suppose that G and S are defined as above. S is quadratically invariant under G if and only if

$$\gamma_p + (C/R) \;\geq\; \gamma_t$$

Proof: Since any norm satisfies the triangle inequality, the pure transmission delays clearly satisfy the triangle inequality, so applying Theorem 6, S is quadratically invariant under G if and only if

$$\gamma_p \|x_i - x_j\| + C \;\geq\; \gamma_t \|x_i - x_j\| \qquad \text{for all } i, j$$

which is equivalent to

$$\gamma_p + (C/R) \;\geq\; \gamma_t$$

■

Thus we see that, in the absence of computational delay, finding the minimum-norm controller may be reduced to a convex optimization problem when the speed of transmission is faster than the speed of propagation; that is, when $\gamma_p \geq \gamma_t$. We also see that this not only remains true in the presence of computational delay, but that we get a buffer relaxing the condition.

A similar result was previously achieved for a very specific case of vehicles equally spaced along a line [1]. This shows how the results of this paper allow us to effortlessly generalize to the case considered in this subsection, where the vehicles have arbitrary positions in arbitrary dimensions. This is a crucial generalization for applications to realistic formation flight problems.

B. Two-Dimensional Lattice Example

In this subsection we will consider subsystems distributed in a lattice, and use these results to derive the conditions for convexity of the associated optimal decentralized control problem.

We first consider the case where the controllers can communicate along the edges of the lattice with a delay of t, and the dynamics similarly propagate along the edges with a delay of p, as illustrated in the left half of Figure 3.

It is a straightforward consequence of this paper that the optimal controllers may be synthesized with convex programming if

$$p \;\geq\; t$$

We now consider a more interesting variant, where the controllers again communicate only along the edges of the lattice, but now the dynamics propagate in all directions, as illustrated in the right half of Figure 3.

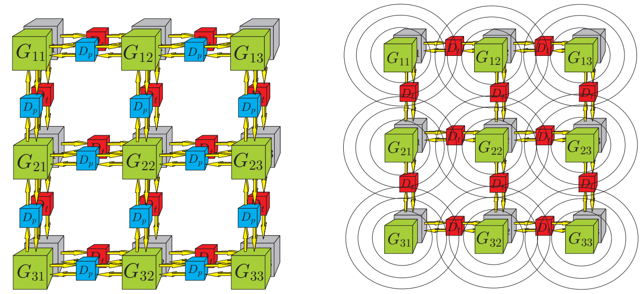

Fig. 3. Two-dimensional lattice with dynamics propagating along edges or in all directions

Let γ_p be the amount of time it takes for the dynamics to propagate one unit of distance. Along a diagonal, for instance, between G_{11} and G_{22}, the propagation delay is $\gamma_p\sqrt{2}$ and the transmission delay is $2t$. The condition for convexity therefore becomes

$$\gamma_p \;\geq\; t\sqrt{2}$$

V. Conclusions

We have studied the problem of finding optimal controllers for multiple subsystems subject to constraints on how quickly they can share information. In Theorem 3 we showed that, presuming the transmission delays satisfy the triangle inequality, if the transmission delay between any pair of subsystems is less than the corresponding propagation delay, then the information constraint is quadratically invariant. This allows for convex synthesis of the optimal decentralized controllers.

We further showed that if we separately account for computational delays, we still find that communicating faster than dynamics propagate along any link allows for convex synthesis of optimal controllers, and in fact, the condition becomes relaxed. We considered an example corresponding to control of vehicles in formation, and showed that optimal controllers may be computed in this manner if the communication speed exceeded the propagation speed, for arbitrary vehicle positions.

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