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Parametrization of All Stabilizing Controllers Subject to Any Structural Constraint

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Abstract— **We consider the problem of designing optimal stabilizing decentralized controllers subject to an arbitrary structural constraint, where each part of the controller has access to some measurements but not others. We develop a parametrization of the stabilizing structured controllers such that the objective is a convex function of the parameter, with the parameter subject to a single quadratic equality constraint.**

I. Introduction

This paper addresses the problem of optimal structured control, where we have multiple controllers, each of which may have access to some sensor measurements but not others. Most conventional controls analysis breaks down when such decentralization is enforced. Finding optimal controllers when different controllers can access different measurements is notoriously difficult even for the simplest such problem [1], and there are results proving computational intractability for the more general case [2], [3].

When a conditional called quadratic invariance holds, which relates these information constraints on the controller to the system being controlled, then the optimal decentralized control problem may be recast as a convex optimization problem [4]. For a particular Youla parametrization, which converts the closed-loop performance objective into a convex function of the new parameter, the information constraint becomes an affine constraint on the parameter, and thus the resulting problem is still convex. The problem of finding the best block diagonal controller however, which represents the case where each subsystem controller may only access measurements from its own subsystem, is never quadratically invariant except for the case where the plant is block diagonal as well; that is, when subsystems do not affect one another.

The parametrization and optimization of stabilizing block diagonal controllers was addressed in a similar fashion using Youla parametrization in [5], focusing on the 2-channel (or 2-block, 2-subsystem, etc.) case. The parametrization similarly converts the objective into a convex function, but the block diagonal constraint on the controller then becomes a quadratic equality constraint on the otherwise free parameter. It is further suggested that the trick for achieving this can be implemented $n-1$ times for n-channel control, resulting in $n-1$ quadratic equality constraints.

While this constraint causes the resulting problem to be nonconvex, it still converts a generally intractable problem into one where the only difficulty is a well-understood type of constraint. Solving this resulting constrained problem is then further explored in [6], and many other methods exist for addressing quadratic equality constraints.

In this paper, we address arbitrary structural constraints, which are not generally block diagonal nor quadratically invariant. We first in Section IV show how to convert this general problem to a block diagonal synthesis problem. In Section V, we then adapt the main insight of [5] and show that this can then be similarly converted into a problem on a stable Youla parameter with a convex objective, but subject to a single quadratic equality constraint, regardless of the number of blocks.

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II. Preliminaries

We suppose that we have a generalized plant $P \in \mathcal{R}_p$ partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & G \end{bmatrix}$$

We define the ***closed-loop map*** by

$$f(P,K) = P_{11} + P_{12}K(I - GK)^{-1}P_{21}$$

The map $f(P,K)$ is also called the (lower) ***linear fractional transformation*** (LFT) of P and K. Note that we abbreviate $G = P_{22}$, since we will refer to that block frequently, and so that we may refer to its subdivisions without ambiguity. This interconnection is shown in Figure 1.

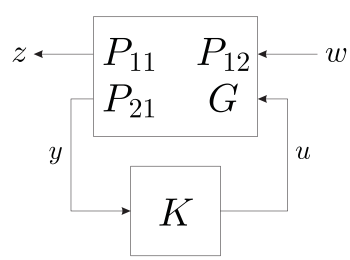

Fig. 1. Linear fractional interconnection of P and K

We suppose that there are n_y sensor measurements and n_u control actions, and thus partition the sensor measurements and control actions as

$$y = \begin{bmatrix} y_1^T & \dots & y_{n_y}^T \end{bmatrix}^T \qquad u = \begin{bmatrix} u_1^T & \dots & u_{n_u}^T \end{bmatrix}^T$$

with partition sizes

$$y_i \in L_2^{p_i} \ \forall\ i \in 1,\dots,n_y \qquad u_j \in L_2^{m_j} \ \forall\ j \in 1,\dots,n_u$$

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and total sizes

$$y \in L_2^p, \quad \sum_{i=1}^{n_y} p_i = p; \qquad u \in L_2^m, \quad \sum_{j=1}^{n_u} m_j = m$$

and then further partition G and K as

$$G = \begin{bmatrix} G_{11} & \dots & G_{1n_u} \\ \vdots & & \vdots \\ G_{n_y1} & \dots & G_{n_yn_u} \end{bmatrix} \qquad K = \begin{bmatrix} K_{11} & \dots & K_{1n_y} \\ \vdots & & \vdots \\ K_{n_u1} & \dots & K_{n_un_y} \end{bmatrix}$$

with block sizes

$$G_{ij} \in \mathcal{R}_{sp}^{p_i \times m_j}, \qquad K_{ji} \in \mathcal{R}_p^{m_j \times p_i} \quad \forall\ i, j$$

and total sizes

$$G \in \mathcal{R}_{sp}^{p \times m} \qquad K \in \mathcal{R}_p^{m \times p}$$

This will typically represent n subsystems, each with its own controller, in which case we will have $n = n_y = n_u$, but this does not have to be the case.

a) Transfer functions.: We use the following standard notation. Denote the imaginary axis by

$$j\mathbb{R} = \big\{ z \in \mathbb{C} \mid \Re(z) = 0 \big\}$$

and the closed right half of the complex plane by

$$\mathbb{C}_+ = \big\{ z \in \mathbb{C} \mid \Re(z) \geq 0 \big\}$$

We define transfer functions for continuous-time systems therefore determined on $j\mathbb{R}$, but we could also define transfer functions for discrete-time systems determined on the unit circle. A rational function $G : j\mathbb{R} \to \mathbb{C}$ is called ***real-rational*** if the coefficients of its numerator and denominator polynomials are real. Similarly, a matrix-valued function $G : j\mathbb{R} \to \mathbb{C}^{m \times n}$ is called real-rational if G_{ij} is real-rational for all i, j. It is called ***proper*** if

$$\lim_{\omega \to \infty} G(j\omega) \quad \text{exists and is finite,}$$

and it is called ***strictly proper*** if

$$\lim_{\omega \to \infty} G(j\omega) = 0.$$

Denote by $\mathcal{R}_p^{m \times n}$ the set of matrix-valued real-rational proper transfer matrices

$$\mathcal{R}_p^{m \times n} = \Big\{ G : j\mathbb{R} \to \mathbb{C}^{m \times n} \mid G \text{ proper, real-rational} \Big\}$$

and let $\mathcal{R}_{sp}^{m \times n}$ be

$$\mathcal{R}_{sp}^{m \times n} = \Big\{ G \in \mathcal{R}_p^{m \times n} \mid G \text{ strictly proper} \Big\}$$

Also let $\mathcal{RH}_\infty$ be the set of real-rational proper stable transfer matrices

$$\mathcal{RH}_\infty^{m \times n} = \Big\{ G \in \mathcal{R}_p^{m \times n} \mid G \text{ has no poles in } \mathbb{C}_+ \Big\}$$

It can be shown that functions in $\mathcal{RH}_\infty$ are determined by their values on $j\mathbb{R}$, and thus we can regard $\mathcal{RH}_\infty$ as a subspace of $\mathcal{R}_p$. If $A \in \mathcal{R}_p^{n \times n}$ we say A is ***invertible*** if $\lim_{\omega \to \infty} A(j\omega)$ is an invertible matrix and $A(j\omega)$ is invertible for almost all $\omega \in \mathbb{R}$. Note that this is different from the definition of invertibility for the associated multiplication operator on L_2. If A is invertible we write $B = A^{-1}$ if $B(j\omega) = A(j\omega)^{-1}$ for almost all $\omega \in \mathbb{R}$.

When the dimensions are implied by context, we omit the superscripts of $\mathcal{R}_p^{m \times n}, \mathcal{R}_{sp}^{m \times n}, \mathcal{RH}_\infty^{m \times n}$.

Let I_n represent the $n \times n$ identity.

A. Stabilization

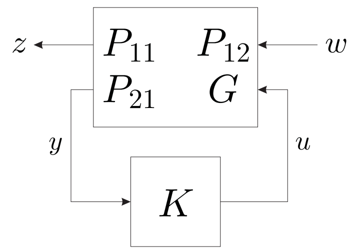

Fig. 2. Linear fractional interconnection of P and K

We say that K ***stabilizes*** P if in Figure 1 the nine transfer matrices from w, v_1, v_2 to z, u, y belong to $\mathcal{RH}_\infty$. We say that K ***stabilizes*** G if in the figure the four transfer matrices from v_1, v_2 to u, y belong to $\mathcal{RH}_\infty$. P is called ***stabilizable*** if there exists $K \in \mathcal{R}_p^{m \times p}$ such that K stabilizes P. The following standard result relates stabilization of P with stabilization of G.

Theorem 1: Suppose $G \in \mathcal{R}_{sp}^{p \times m}$ and $P \in \mathcal{R}_p^{(n_z+p) \times (n_w+m)}$, and suppose P is stabilizable. Then K stabilizes P if and only if K stabilizes G.

Proof: See, for example, Chapter 4 of [7]. ■

For a given system P, all controllers that stabilize the system may be parameterized using the well-known Youla parametrization [8], stated below.

Theorem 2: Suppose that we have a doubly coprime factorization of G over $\mathcal{RH}_\infty$, that is, $M_l, N_l, X_l, Y_l, M_r, N_r, X_r, Y_r \in \mathcal{RH}_\infty$ such that $G = N_r M_r^{-1} = M_l^{-1} N_l$ and

$$\begin{bmatrix} X_l & -Y_l \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & Y_r \\ N_r & X_r \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}. \tag{1}$$

Then the set of all stabilizing controllers is given by

$$\begin{aligned} &\{K \in \mathcal{R}_p \mid K \text{ stabilizes } G\} \\ &\qquad = \Big\{ (Y_r - M_r Q)(X_r - N_r Q)^{-1} \mid \\ &\qquad\qquad X_r - N_r Q \text{ is invertible}, Q \in \mathcal{RH}_\infty \Big\} \\ &\qquad = \Big\{ (X_l - Q N_l)^{-1}(Y_l - Q M_l) \mid \\ &\qquad\qquad X_l - Q N_l \text{ is invertible}, Q \in \mathcal{RH}_\infty \Big\}. \end{aligned}$$

Furthermore, the set of all closed-loop maps achievable with stabilizing controllers is

$$\begin{aligned} &\Big\{ f(P, K) \;\Big|\; K \in \mathcal{R}_p,\ K \text{ stabilizes } P \Big\} \\ &= \Big\{ T_1 - T_2 Q T_3 \;\Big|\; X_r - N_r Q \text{ is invertible},\ Q \in \mathcal{RH}_\infty \Big\}, \end{aligned} \tag{2}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$ are given by

$$\begin{aligned} T_1 &= P_{11} + P_{12} Y_r M_l P_{21} \\ T_2 &= P_{12} M_r \\ T_3 &= M_l P_{21}. \end{aligned} \tag{3}$$

Proof: See, for example, Chapter 4 of [7]. ■

The parameter Q is usually referred to as the Youla parameter. We now show that the two parametrizations above give the same change of variables.

Lemma 3: Suppose $G \in \mathcal{R}_{sp}$. Then the set of all stabilizing controllers is given by

$$\begin{aligned} &\{K \in \mathcal{R}_p \mid K \text{ stabilizes } P\} = \\ &\Big\{(Y_r - M_r Q)(X_r - N_r Q)^{-1} = (X_l - Q N_l)^{-1}(Y_l - Q M_l) \\ &\qquad\qquad \mid Q \in \mathcal{RH}_\infty\Big\}. \end{aligned}$$

Proof: The equivalence of stabilizing G and the generalized plant P follows from Theorem 1. It follows from G strictly proper that N_l, N_r are strictly proper, and thus the invertibility conditions of Theorem 2 are always satisfied. It just remains to show that the two parametrizations give the same mapping from parameter to controller. This follows very similarly from [5]:

$$\begin{aligned} &(Y_r - M_r Q_r)(X_r - N_r Q_r)^{-1} \\ &\qquad = (X_l - Q_l N_l)^{-1}(Y_l - Q_l M_l) \\ \Leftrightarrow\ &(X_l - Q_l N_l)(Y_r - M_r Q_r) \\ &\qquad = (Y_l - Q_l M_l)(X_r - N_r Q_r) \\ \Leftrightarrow\ &\underbrace{(X_l Y_r - Y_l X_r)}_{0} + Q_l \underbrace{(M_l X_r - N_l Y_r)}_{I_p} \\ &\quad = \underbrace{(X_l M_r - Y_l N_r)}_{I_m} Q_r + Q_l \underbrace{(M_l N_r - N_l M_r)}_{0} Q_r \\ \Leftrightarrow\ &\qquad Q_l = Q_r. \end{aligned}$$

■

Remark 4: Even if G is not strictly proper, the invertibility conditions still hold for almost all parameters Q [9, p.111].

III. Problem Formulation

We develop the optimization problem that we need to solve for optimal synthesis subject to an arbitrary structural constraint.

A. Structural Constraints

Structural constraints, which specify that each controller may access certain sensor measurements but not others, manifest themselves as sparsity constraints on the controller to be designed. We here introduce some notation for representing this type of constraint.

Let $\mathbb{B} = \{0, 1\}$ represent the set of binary numbers. Suppose $A^{\text{bin}} \in \mathbb{B}^{m \times n}$ is a binary matrix. We define the subspace

$$\begin{aligned} \text{Sparse}(A^{\text{bin}}) = \Big\{B \in \mathcal{R}_p \mid B_{ij}(j\omega) = 0 \text{ for all } i, j \\ \text{such that } A^{\text{bin}}_{ij} = 0 \text{ for almost all } \omega \in \mathbb{R}\Big\} \end{aligned}$$

giving all of the proper transfer function matrices which satisfy the given sparisty constraint.

We then represent the constraints on the overall controller with a binary matrix $K^{\text{bin}} \in \mathbb{B}^{n_u \times n_y}$ where

$$K^{\text{bin}}_{kl} = \begin{cases} 1, & \text{if control input } k \\ & \text{may access sensor measurement } l \\ 0, & \text{if not.} \end{cases}$$

The subspace of admissible controllers is then given as

$$S = \text{Sparse}(K^{\text{bin}}).$$

We can now set up our main problem of finding the best such controller.

B. Problem Setup

Given a generalized plant P and a subspace of admissible controllers S, we would then like to solve the following problem:

$$\begin{aligned} \text{minimize} \quad & \|f(P, K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned} \tag{4}$$

Here $\|\cdot\|$ is any norm on the closed-loop map chosen to encapsulate the control performance objectives. The subspace of admissible controllers, S, has been defined above to encapsulate the constraints on which controllers can access which sensor measurements. We call the subspace S the ***information constraint***.

Many decentralized control problems may be expressed in the form of problem (4). In this paper, we focus on the case where S is defined by structural constraints as discussed above.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved with many standard techniques. Note that the cost function $\|f(P, K)\|$ is in general a non-convex function of K. If the information constraint is quadratically invariant [4] with respect to the plant, then problem may be recast as a convex optimization problem, but no computationally tractable approach is known for solving this problem for arbitrary P and S.

IV. Diagonalization

In this section, we show how the problem of finding the optimal structured controller $K \in S$ can be converted to a problem of finding an optimal block diagonal controller.

Similar equivalences of decentralization constraints were considered for a different purpose in [10], and we use consistent notation where possible.

The concepts in this section are fairly straightforward, but covering the general case rigorously is unfortunately of a tedious nature requiring multiple subscripts. The reader wishing to skim this section need only take away that the plant may be altered such that sensor measurements are repeated and then controller inputs reconstituted in such a way that the active parts of the controller are shuffled around to make a block diagonal controller, and such that the closed-loop map remains the same. Example 5 illustrates the shuffling for a particular controller structure.

First, given structural constraints proscribed by a binary matrix K^{bin}, define the total number of active blocks in an admissible controller as

$$a = \sum_{k=1}^{n_u} \sum_{l=1}^{n_y} K_{kl}^{\text{bin}}.$$

For each active block, that is, for each pair (k,l) such that $K_{kl}^{\text{bin}} = 1$, assign a unique $\alpha \in \{1,\ldots,a\}$, and let $\kappa_\alpha = k$ and $\lambda_\alpha = l$. To avoid the use of triple subscripts later, we also index the block sizes of our diagonal controller as $b_\alpha = m_{\kappa_\alpha}$ and $c_\alpha = p_{\lambda_\alpha}$.

If we construct a block diagonal controller with the active blocks along the diagonal, it will then have a total output size and input size, respectively, of

$$a_u = \sum_{\alpha=1}^{a} b_\alpha, \qquad a_y = \sum_{\alpha=1}^{a} c_\alpha.$$

We then define the subspace of block diagonal controllers $\tilde{S} \subset \mathcal{R}_p^{a_u \times a_y}$ as

$$\tilde{S} = \left\{ \text{diag}(\tilde{K}_{11},\ldots,\tilde{K}_{aa}) \mid \tilde{K}_{\alpha\alpha} \in \mathcal{R}_p^{b_\alpha \times c_\alpha} \ \forall\ \alpha \right\}.$$

We now wish to show that we can construct a problem with the objective of finding the optimal stabilizing block-diagonal $\tilde{K} \in \tilde{S}$ which is equivalent to our original problem of finding the optimal stabilizing structured $K \in S$.

Let's first consider the sensor measurements that such a controller would have access to and call this $\tilde{y}$. An active block K_{kl} is able to see y_l, and so the corresponding part of $\tilde{y}_\alpha$ has to be a copy of y_l. $\tilde{y}$ thus becomes an extended version of the original vector of sensor measurements y, with measurements repeated as necessary for each active block of the controller which has access to it. We can then define $\tilde{y} \in L_2^{a_y}$ as follows

$$\tilde{y} = \begin{bmatrix} \tilde{y}_1^T & \cdots & \tilde{y}_a^T \end{bmatrix}^T \qquad \text{where} \qquad \tilde{y}_\alpha = y_{\lambda_\alpha}$$

We can map from y to $\tilde{y}$ by defining the matrix $U \in \mathbb{R}^{a_y \times p}$ as

$$U_{\alpha l} = \begin{cases} I_{p_l}, & \text{if } \lambda_\alpha = l \\ 0, & \text{otherwise.} \end{cases}$$

Since each sensor measurement is accessed by at least one controller block (otherwise it wouldn't be a sensor measurement), U has full column rank, and we can define a left inverse $U^\dagger \in \mathbb{R}^{p \times a_y}$ as

$$U^\dagger = (U^T U)^{-1} U^T.$$

We then have $\tilde{y} = Uy$ and $y = U^\dagger \tilde{y}$.

We next turn our attention to the output of the block diagonal controller, $\tilde{u} = \tilde{K}\tilde{y}$. This will comprise the output of each active controller block, from which we will have to reconstitute the original controller input u. They are related as follows

$$\begin{aligned} u_k &= \sum_{l=1}^{n_y} K_{kl} y_l \\ &= \sum_{\alpha:\kappa_\alpha = k} \tilde{K}_\alpha \tilde{y}_\alpha \\ &= \sum_{\alpha:\kappa_\alpha = k} \tilde{u}_\alpha \end{aligned}$$

and so we can map from $\tilde{u}$ to u by defining the matrix $V \in \mathbb{R}^{m \times a_u}$ as

$$V_{k\alpha} = \begin{cases} I_{m_k}, & \text{if } \kappa_\alpha = k \\ 0, & \text{otherwise.} \end{cases}$$

Since each controller input can see at least one sensor measurement, V has full row rank, and we can define a right inverse $V^\dagger \in \mathbb{R}^{a_u \times m}$ as

$$V^\dagger = V^T (VV^T)^{-1}.$$

We then have $u = V\tilde{u}$, but do not necessarily have $\tilde{u} = V^\dagger u$.

We can now define a new generalized plant $\tilde{P} \in \mathcal{R}_p^{(n_z + a_y) \times (n_w + a_u)}$ with the following components

$$\begin{aligned} \tilde{P}_{11} &= P_{11} & \tilde{P}_{12} &= P_{12} V \\ \tilde{P}_{21} &= U P_{21} & \tilde{G} &= UGV \end{aligned} \tag{5}$$

which maps $(w, \tilde{u}) \to (z, \tilde{y})$.

Example 5: Suppose we are trying to find the best controller $K \in S$ where $S = \text{Sparse}(K^{\text{bin}})$ and where the admissible controller structure is given by

$$K^{\text{bin}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

that is, we need to find the best 3×4 controller where only 5 particular parts of the controller may be active.

We then have $a = 5$, and assign $(\kappa_1, \lambda_1) = (1,1)$, $(\kappa_2, \lambda_2) = (3,1)$, $(\kappa_3, \lambda_3) = (2,2)$, $(\kappa_4, \lambda_4) = (1,3)$, and $(\kappa_5, \lambda_5) = (3,4)$. We then get

$$U = \begin{bmatrix} I & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \qquad V = \begin{bmatrix} I & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & I \end{bmatrix}$$

such that $\tilde{y} = Uy = \begin{bmatrix} y_1^T & y_1^T & y_2^T & y_3^T & y_4^T \end{bmatrix}^T$ and such that $u = V\tilde{u} = \begin{bmatrix} (\tilde{u}_1 + \tilde{u}_4)^T & \tilde{u}_3^T & (\tilde{u}_2 + \tilde{u}_5)^T \end{bmatrix}^T$. The matrix U repeats the first sensor measurement since the first two active parts of the controller, and thus the first two parts of the block diagonal controller, both need to access it, and then the matrix V takes the 5 signals from the block diagonal controller ($\tilde{u}$) and reconstitutes the 3 controller inputs (u).

We may then replace the given generalized plant P with $\tilde{P}$ as in (5), and any admissible controller $K \in S$ by the block diagonal controller $\tilde{K} = \mathrm{diag}(K_{\kappa_1\lambda_1}, \ldots, K_{\kappa_5\lambda_5}) = \mathrm{diag}(K_{11}, K_{31}, K_{22}, K_{13}, K_{34})$ such that $K = V\tilde{K}U$, and this maintains the same closed-loop map.

Having motivated these transformations, we now need to show that they indeed yield a diagonal synthesis problem which is equivalent to our original problem.

We first give a lemma which verifies that we have properly set up our bijection from admissible structured controllers $K \in S$ to block diagonal controllers $\tilde{K} \in \tilde{S}$.

Lemma 6: Given $\tilde{K} \in \tilde{S}$, we can let $K = V\tilde{K}U$, and then $K \in S$.

Given $K \in S$, we can let

$$\tilde{K}_{\alpha\beta} = \begin{cases} K_{kl}, & \text{if } \alpha = \beta, k = \kappa_\alpha, l = \lambda_\alpha \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

and then $\tilde{K} \in \tilde{S}$ and $K = V\tilde{K}U$.

Proof: First, given $\tilde{K} \in \tilde{S}$, let $K = V\tilde{K}U$. Then,

$$\begin{aligned} K_{kl} &= \sum_{\alpha=1}^{a}\sum_{\beta=1}^{a} V_{k\alpha}\tilde{K}_{\alpha\beta}U_{\beta l} \\ &= \sum_{\alpha=1}^{a} V_{k\alpha}\tilde{K}_{\alpha\alpha}U_{\alpha l} \quad \text{since } \tilde{K} \text{ diagonal} \\ &= \begin{cases} \tilde{K}_{\alpha\alpha}, & \text{if } \exists\ \alpha \text{ such that } (k,l) = (\kappa_\alpha, \lambda_\alpha) \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \tilde{K}_{\alpha\alpha}, & \text{if } K^{\text{bin}}_{kl} = 1,\ (k,l) = (\kappa_\alpha, \lambda_\alpha) \\ 0, & \text{if } K^{\text{bin}}_{kl} = 0 \end{cases} \end{aligned}$$

and so $K \in S$.

Now, given $K \in S$, we can define $\tilde{K}$ as in (6). $\tilde{K} \in \tilde{S}$ clearly as it is defined to be block diagonal, and then $K = V\tilde{K}U$ is verified by the above equalities. ■

Remark 7: Note that given $K \in S$, $V^\dagger K U^\dagger$ is generally not in $\tilde{S}$.

If $K = V\tilde{K}U$, then

$$\begin{aligned} f(P,K) &= P_{11} + P_{12}(V\tilde{K}U)(I - GV\tilde{K}U)^{-1}P_{21} \\ &= P_{11} + (P_{12}V)\tilde{K}(I - UGV\tilde{K})^{-1}UP_{21} \\ &= \tilde{P}_{11} + \tilde{P}_{12}\tilde{K}(I - \tilde{G}\tilde{K})^{-1}\tilde{P}_{21} \\ &= f(\tilde{P}, \tilde{K}) \end{aligned}$$

where we used the push-through identity in the second step, and thus the closed-loop maps are identical.

With repeated use of the push-through identity we also find

$$\begin{bmatrix} I & K \\ G & I \end{bmatrix}^{-1} = \begin{bmatrix} V & 0 \\ 0 & U^\dagger \end{bmatrix} \begin{bmatrix} I & \tilde{K} \\ \tilde{G} & I \end{bmatrix}^{-1} \begin{bmatrix} V^\dagger & 0 \\ 0 & U \end{bmatrix}$$

Thus if $\tilde{K}$ stabilizes $\tilde{G}$, then K stabilizes G. The converse does not generally follow from this relation, but if an unstable mode is suppressed by $V, U^\dagger, V^\dagger$, or U, we just need a stabilizing $\tilde{K}$ which yields $V\tilde{K}(I - \tilde{G}\tilde{K})^{-1}U = K(I - GK)^{-1}$ to achieve the same closed-loop map. Generalizing the technical conditions needed for this, or understanding in what manner enforcing internal stability on the diagonalized problem represents a stronger notion of stability enforced on the orignal problem, is ongoing work.

V. Parametrization

In this section, we assume that our problem has been converted to one of finding an optimal block diagonal controller, and address the problem of parametrizing all of the stabilizing controllers and all of the achievable closed-loop maps, to further transform the optimization problem to one over a stable parameter with a convex objective. We show that this can be achieved, with the diagonalization manifesting itself as a quadratic equality constraint on the parameter.

The key insight in this section, that a block diagonal constraint can be expressed as in (7), which then becomes a quadratic constraint in the Youla parameter, is largely derived from Manousiouthakis [5]. There this idea was introduced for 2-channel control (block diagonal with 2 blocks), and it was suggested that the same technique could be used $n-1$ times, resulting in $n-1$ quadratic equality constraints on the Youla parameter, to enforce a block diagonal constraint with n blocks. Here, in addition to having first started with an arbitrary structural constraint, we now show how this parameterization can be achieved with just one quadratic equality constraint, regardless of the number of blocks.

Define $L_l \in \mathbb{R}^{a_u \times a_u}$ as

$$L_l = \mathrm{diag}(I_{b_1}, 2I_{b_2}, \ldots, aI_{b_a})$$

and define $L_r \in \mathbb{R}^{a_y \times a_y}$ as

$$L_r = \mathrm{diag}(I_{c_1}, 2I_{c_2}, \ldots, aI_{c_a})$$

and note that the two matrices are identical if all controller blocks are square or scalar.

We now show how these matrices can be used to enforce a block diagonal constraint.

Lemma 8: Given $\tilde{K} \in \mathcal{R}_p^{a_u \times a_y}$,

$$L_l\tilde{K} = \tilde{K}L_r \qquad \Leftrightarrow \qquad \tilde{K} \in \tilde{S}. \tag{7}$$

Proof:

$$\begin{aligned} & L_l\tilde{K} = \tilde{K}L_r \\ \Leftrightarrow \quad & \sum_{k=1}^{a}(L_l)_{ik}\tilde{K}_{kj} = \sum_{k=1}^{a}\tilde{K}_{ik}(L_r)_{kj} \quad \forall\ i,j \\ \Leftrightarrow \quad & (L_l)_{ii}\tilde{K}_{ij} = \tilde{K}_{ij}(L_r)_{jj} \quad \forall\ i,j \text{ since } L_l, L_r \text{ diag.} \\ \Leftrightarrow \quad & i\tilde{K}_{ij} = j\tilde{K}_{ij} \quad \forall\ i,j \\ \Leftrightarrow \quad & \tilde{K}_{ij} = 0 \quad \forall\ i \neq j \\ \Leftrightarrow \quad & \tilde{K} \in \tilde{S}. \end{aligned}$$

■

The following theorem is the main result of this section. It shows that all of the stabilizing block diagonal controllers can be parametrized by a stable Youla parameter, subject to a single quadratic equality constraint. It further shows that the set of all achievable closed-loop maps may then be expressed an an affine function of this Youla parameter, subject to the same quadratic equality constraint.

Theorem 9: Suppose that $\tilde{P}$ is stabilizable, and that we have a doubly coprime factorization of $\tilde{G}$ as in (1). Then

$$\{\tilde{K} \in \mathcal{R}_p \mid \tilde{K} \text{ stabilizes } \tilde{P},\ \tilde{K} \in \tilde{S}\} = \{(Y_r - M_r Q)(X_r - N_r Q)^{-1} \mid Q \in \mathcal{RH}_\infty,\ q(Q) = 0\}$$

where

$$q(Q) = \begin{bmatrix} I_{a_u} & Q \end{bmatrix} \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \begin{bmatrix} I_{a_y} \\ Q \end{bmatrix} \tag{8}$$

and where

$$\begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} = \begin{bmatrix} X_l & Y_l \\ -N_l & -M_l \end{bmatrix} \begin{bmatrix} L_l & 0 \\ 0 & L_r \end{bmatrix} \begin{bmatrix} Y_r & -M_r \\ -X_r & N_r \end{bmatrix}. \tag{9}$$

Further,

$$\{f(\tilde{P}, \tilde{K}) \mid \tilde{K} \text{ stabilizes } \tilde{P},\ \tilde{K} \in \tilde{S}\} = \{T_1 - T_2 Q T_3 \mid Q \in \mathcal{RH}_\infty,\ q(Q) = 0\} \tag{10}$$

where T_i are given as in (3).

Proof: With this change of variables, the equivalence of $Q \in \mathcal{RH}_\infty$ with $\tilde{K}$ stabilizing $\tilde{P}$, as well as $f(\tilde{P}, \tilde{K}) = T_1 - T_2 Q T_3$, follow from Theorem 2. It just remains to show that the diagonal constraint on the controller is equivalent to the quadratic constraint on the Youla parameter. Note that we utilize Lemma 3 and the equivalence of the left and right parametrizations in the second step:

$$\begin{aligned}
& \tilde{K} \in \tilde{S} \\
\Leftrightarrow\ & L_l \tilde{K} = \tilde{K} L_r \\
\Leftrightarrow\ & L_l (Y_r - M_r Q)(X_r - N_r Q)^{-1} \\
& \qquad = (X_l - Q N_l)^{-1} (Y_l - Q M_l) L_r \\
\Leftrightarrow\ & (X_l - Q N_l) L_l (Y_r - M_r Q) \\
& \qquad = (Y_l - Q M_l) L_r (X_r - N_r Q) \\
\Leftrightarrow\ & \underbrace{(X_l L_l Y_r - Y_l L_r X_r)}_{W_1} + \underbrace{(Y_l L_r N_r - X_l L_l M_r)}_{W_2} Q \\
& + Q \underbrace{(M_l L_r X_r - N_l L_l Y_r)}_{W_3} + Q \underbrace{(N_l L_l M_r - M_l L_r N_r)}_{W_4} Q = 0 \\
\Leftrightarrow\ & q(Q) = 0
\end{aligned}$$

■

We may thus solve the following equivalent problem

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2 Q T_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & q(Q) = 0 \end{aligned} \tag{11}$$

to find the optimal Q^*, recover the optimal diagonal controller as $\tilde{K}^* = (Y_r - M_r Q^*)(X_r - N_r Q^*)^{-1}$, and then recover the optimal structured controller for our original problem (4) as $K^* = V \tilde{K}^* U$.

Remark 10: Note that the calculation of the Youla parameters (1) and the closed-loop parameters (3) in Theorem 9 must be based on $\tilde{G}$ and $\tilde{P}$.

Remark 11: If the quadratic term, W_4, is 0, then the constraint if affine, and the resulting optimization problem is convex. One can show that this occurs if and only if $\tilde{G}$ is block diagonal, which corresponds to a special case of quadratic invariance.

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VI. CONCLUSIONS

We have considered the problem of synthesizing optimal stabilizing controllers subject to an arbitrary structural constraint. We first showed how to recast this as a block diagonal synthesis problem, and then how to recast that as a problem over a stable Youla parameter with a convex objective.

The general problem we are addressing is known to be intractable, and so it is not surprising that the resulting optimization problem is not convex in general. However, we have taken a broad class of important intractable problems, shown how to handle them in a unified manner, and shown how the inherent difficulty of the problem can be concentrated into a single quadratic equality constraint.

The synthesis of optimal (decentralized) control via Youla parametrization can now be summarized as follows. Without structural constraints, finding the optimal stabilizing controller can be cast as optimizing a convex function of the Youla parameter, where the parameter is free and stable. If the controller is instead subject to a quadratically invariant structural constraint, the parameter is subject to an affine equality constraint. If the controller is subject to a structural constraint which is not quadratically invariant, the parameter is subject to a quadratic equality constraint.

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