53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Stabilizing Decentralized Systems with Arbitrary Information Structure

Alborz Alavian and Michael Rotkowitz

***Abstract*— A seminal result in decentralized control is the development of fixed modes by Wang and Davison in 1973 - that plant modes which cannot be moved with a static decentralized controller cannot be moved by a dynamic one either, and that the other modes which can be moved can be shifted to any chosen location with arbitrary precision. These results were developed for perfectly decentralized, or block diagonal, information structure, where each control input may only depend on a single corresponding measurement. Furthermore, the results were claimed after a preliminary step was demonstrated, omitting a rigorous induction for each of these results, and the remaining task is nontrivial.**

In this paper, we consider fixed modes for arbitrary information structures, where certain control inputs may depend on some measurements but not others. We provide a comprehensive proof that the modes which can be altered by a static controller with the given structure can be moved by a dynamic one to any chosen location with arbitrary precision, thus generalizing and solidifying the second part of Wang and Davison's result. A previous paper discussed the first part.

This shows that a system can be stabilized by a linear time-invariant controller with the given information structure as long as all of the modes which are fixed with respect to that structure are in the left half-plane; an algorithm for synthesizing such a stabilizing decentralized controller is then distilled from the proof.

I. Introduction

This paper is concerned with the stabilization of decentralized control systems, for which certain controller inputs may depend on some measurements but not others. This corresponds to finding a stabilizing controller which satisfies a given sparsity constraint. A special case of this, sometimes referred to as *perfectly decentralized control*, occurs when each control input may depend only on a single associated measurement, which corresponds to finding a stabilizing controller which is (block) diagonal.

This special case is sometimes itself referred to as *decentralized control*, particularly in the literature from a few decades ago. This malleability or evolution of the definition has not only caused some confusion, but has also resulted in some important results in the field only being studied for this special case.

We assume that plants and controllers are finite-dimensional, linear time-invariant (FDLTI), except for when we say otherwise.

A seminal result in decentralized control is the development of *fixed modes* by Wang and Davison in 1973 [1]. This paper studied (FDLTI) perfectly decentralized stabilization of FDLTI systems. Its contributions can be broken into three main components - a definition establishing the framework, and two subsequent results. Fixed modes were defined as those modes of the plant which could not be altered by any static perfectly decentralized controller (that is, by any diagonal matrix). The first result was that these fixed modes could also not be altered by any dynamic perfectly decentralized controller; if you can't move it with a static diagonal controller, you can't move it with a dynamic diagonal controller. The second result was that if a mode is not fixed, then it can be moved arbitrarily close to any chosen location in the complex plane (provided that it has a complex conjugate pair if it is not real). These can be taken together to state that a system is stabilizable by a (dynamic) perfectly decentralized controller if and only if all of its (static) fixed modes are in the left half-plane (LHP).

When proving these results, it was shown that allowing one part of the controller to be dynamic does not result in any fewer fixed modes than a static controller, and then claimed that the second component followed; that is, that a dynamic controller would not be able to move any of the fixed modes. Similarly, it was shown that a single non-fixed mode could be moved to any chosen location, and then claimed that the third component followed; that is, that an arbitrary number of non-fixed modes could be simultaneously moved to chosen locations by a single controller. Getting from these initial steps to a rigorous inductive argument, however, is not trivial.

We seek to study these fundamental concepts for arbitrary information structure, while developing robust notation and rigorous proofs, thus placing the new and existing results on a sound mathematical footing.

We first introduced notation for fixed modes that allows it to vary with information structure, as well as with the type of controllers allowed (static, dynamic, linear, etc.). We then showed in a preceding paper [3] that, for arbitrary information structure, the fixed modes with respect to dynamic controllers are the same as the fixed modes with respect to static controllers, thus extending and solidifying part of the seminal results of Wang and Davison. In this paper, we provide a rigorous proof that the non-fixed modes can then be moved to within an arbitrarily small distance of chosen locations, using a dynamic LTI controller with the given structure, thus extending and solidifying the remaining part of their work. The proof is constructive, and we lastly distill an explicit algorithm for the stabilizing decentralized controller synthesis from the proof.

The obvious potential benefits of this work are an increased understanding of decentralized stabilizability, and the verification of important existing results. It is also our hope that the notation developed will be useful in further extending our understanding of decentralized stabilizability to richer classes of controllers for which the fixed modes may diminish relative to the original static definition, particularly non-linear and/or time-varying controllers [4]–[7]. We further note that demon-

A. Alavian is with the Department of Electrical and Computer Engineering, University of Maryland, College Park, MD 20742 USA, alavian@umd.edu.

M. C. Rotkowitz is with the Institute for Systems Research and the Department of Electrical and Computer Engineering, The University of Maryland, College Park, MD 20742 USA, mcrotk@umd.edu.

strating the results of this paper directly for arbitrary structure, as opposed to attempting to diagonalize the problem and then prove the original perfectly decentralized results, would likely be useful when other types of stability are required which are not invariant under such transformations, though we currently focus on internal state stability. Finally, while the proofs in [1], (as well as [7]), are constructive in nature, they do not clearly lead to an explicit synthesis algorithm. A further advantage of proving this result in earnest was the ability to extract such an algorithm, which then finds a stabilizing LTI decentralized controller whenever one exists. We later learned of the work in [2], which also contains an algorithm; we will briefly contrast them when discussing an example.

The organization of this paper is as follows. In Section II we define notation and preliminaries, including our definition of fixed modes and the controller types that we will later need. In Section III we review certain previous results. In Section IV, we then state and prove our main results. In Section V, we give the explicit computational algorithm, along with a numerical example, followed by some concluding remarks in Section VI.

II. Preliminaries

We proceed with the following preliminary definitions. Let $\Re(\cdot)$ denote the real part of any complex number. Define $\mathbb{C}$ to be the complex plane and $\mathbb{C}^- \triangleq \{\lambda \in \mathbb{C} | \Re(\lambda) < 0\}$ to be the open left-half plane, and $\mathbb{C}^+ \triangleq \mathbb{C} \setminus \mathbb{C}^- = \{\lambda \in \mathbb{C} | \Re(\lambda) \geq 0\}$, to be the closed right-half plane. Let $\mathbf{e}_i$ denote the unit vector of all zeros except for i^{th} element which is 1. The $\|\cdot\|_\infty$ for real matrices is defined as:

$$\|A\|_\infty = \max_i \Big(\sum\nolimits_j |A_{ij}| \Big)$$

We consider an FDLTI plant $P(\sigma)$ (where $\sigma = s, z$ depending on whether we are considering continuous or discrete cases, we use σ for statements that apply to both continuous time and discrete time cases). A state space representation of P is denoted by (A_P, B_P, C_P, D_P). All controllers under consideration in this paper will also be FDLTI. We will be imposing information constraints on our controller, which manifest as sparsity constraints. We suppose that we have a collection of pairs of indices (i,j) such that the ith input to the plant is allowed to depend on the jth measurement. We then define $\mathcal{S}$, our set of controllers with the desired structure, as those for which $K_{ij} = 0$ for any index pair which is not in this set. For a sparsity pattern $\mathcal{S}$, we similarly let $\text{Adm}(\mathcal{S})$ denote the set of admissible indices for which the controller is allowed to be non-zero, i.e., $(i,j) \notin \text{Adm}(\mathcal{S})$ if and only if $K_{ij} = 0$ for all $K \in \mathcal{S}$. For the ease of notation let K^{bin} be the binary matrix of same size as K, for which $K^{\text{bin}}_{ij} = 1$ if and only if $(i,j) \in \text{Adm}(\mathcal{S})$, and 0 otherwise. Also for simplicity we define $\mathcal{S}_{\text{c}}$ to be the centralized sparsity patterns, i.e., no sparsity ($\text{Adm}(\mathcal{S}_{\text{c}}) = \{(i,j) \;\; \forall\, i,j\}$).

We also define type of a controller that will help us to easily refer to whether a controller K is static, dynamic, or static for some element but dynamic for others. Let the type $\mathcal{T}$ of controller K be defined as follows:

- $\mathcal{T}^{\text{d}}$: Set of finite order dynamic controllers, i.e., A_K, B_K, C_K, D_K each are real matrices of appropriate dimension.
- $\mathcal{T}^{\text{s}}$: Set of static controllers, i.e., A_K, B_K, and C_K are all zero and only $D_K \in \mathbb{R}^{n_u \times n_y}$ could be non-zero, where n_u, and n_y are respectively number of inputs, and outputs of the plant.
- $\mathcal{T}^{\text{s+1}}_{i^+,j^+}$: Set of controllers such that all the elements of controller are static except for $(i^+,j^+)^{\text{th}}$ element which could be dynamic, i.e., for all $(i,j) \neq (i^+,j^+)$, $K_{ij} \in \mathbb{R}$ and K_{i^+,j^+} is a proper transfer function in σ. This could be read as "static plus one".

For any information structure $\mathcal{S}$, let $a \triangleq |\text{Adm}(\mathcal{S})|$ be the number of admissible non-zero indices in controller, and let the tuple $I \triangleq \{(i_1,j_1), \cdots, (i_a,j_a)\}$ be any arbitrary ordering of admissible non-zero indices of controller. Also for any $D \in \mathcal{T}^{\text{s}} \cap \mathcal{S}$, define the following sequence of matrices D_m, $m \in \{0,1,\cdots,a\}$ as:

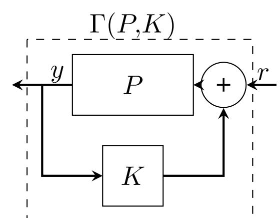
$$D_0 \triangleq 0, \quad D_m \triangleq \sum_{l=1}^{m} \mathbf{e}_{i_l} D_{i_l j_l} \mathbf{e}_{j_l}^T, \quad m \in \{1,\cdots,a\} \tag{1}$$

where $\mathbf{e}_{i_l} \in \mathbb{R}^{n_u}$, and $\mathbf{e}_{j_l} \in \mathbb{R}^{n_y}$, for $p \in \{1,\cdots,a\}$.

Throughout the rest of this paper we assume that $D_P = 0$, and thus the closed-loop has a state-space representation with dynamics matrix denoted by $A_{\text{CL}}(P,K)$, given by:

$$A_{\text{CL}}(P,K) = \begin{pmatrix} A_P + B_P D_K C_P & B_P C_K \\ B_K C_P & A_K \end{pmatrix} \tag{2}$$

as illustrated in Figure 1, let $\Gamma(P,K)$ denote the map from the set-point to the outputs of P (i.e., from r to y), when K is closed around P:



(a) Graphical representation

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{A_{\text{CL}}(P,K)} & \begin{matrix} B_P \\ 0 \end{matrix} \\ \hline C_P & 0 & 0 \end{array}\right](\sigma)$$

(b) A state-space representation

Fig. 1: The map $\Gamma(P,K)$ from set-points to outputs when K is closed around P.

The following property of $\Gamma(\cdot,\cdot)$ can be verified by writing the state-space representation of both sides, we have:

$$\Gamma(\Gamma(P,K_1),K_2) \quad = \quad \Gamma(P, K_1 + K_2). \tag{3}$$

Definition 1: The set of fixed-modes of a plant P with respect to a sparsity pattern $\mathcal{S}$ and a type $\mathcal{T}$, is defined to be:

$$\Lambda(P,\mathcal{S},\mathcal{T}) \triangleq \{\lambda \in \mathbb{C} | \lambda \in \text{eig}(A_{\text{CL}}(P,K)), \forall K \in \mathcal{S} \cap \mathcal{T}\} \tag{4}$$

Remark 2: This reduces to the definition of fixed modes in [1] if $\mathcal{S} = \mathcal{S}_{\text{d}}$ and $\mathcal{T} = \mathcal{T}^{\text{s}}$.

For any FDLTI system P, denote its open-loop modes by $\Lambda(P)$, and for each mode $\lambda \in \Lambda(P)$, let $\mu(\lambda, P)$ denote its multiplicity. We can partition open-loop modes as:

$$\Lambda(P) = \Lambda(P,\mathcal{S},\mathcal{T}^{\text{s}}) \cup \Lambda_+(P) \cup \Lambda_-(P) \tag{5}$$

$$\begin{aligned} \Lambda_+(P) &= \{\alpha \in \Lambda(P) | \Re(\alpha) \geq 0\} \setminus \Lambda(P,\mathcal{S},\mathcal{T}^{\text{s}}) \\ \Lambda_-(P) &= \{\beta \in \Lambda(P) | \Re(\beta) < 0\} \setminus \Lambda(P,\mathcal{S},\mathcal{T}^{\text{s}}) \end{aligned} \tag{6}$$

are respectively distinct unstable (stable) non-fixed open-loop eigenvalues of P. Denote the total (with multiplicities) number of unstable non-fixed modes of a plant P by $\nu(P) \triangleq \sum_{\alpha \in \Lambda_+(P)} \mu(\alpha,P)$.. With a slight abuse of notation, we may use $\Lambda(A)$, $\Lambda_+(A)$, and $\Lambda_-(A)$, for a matrix A, to respectively refer to $\mathrm{eig}(A)$, $\mathrm{eig}(A) \cap \mathbb{C}^+$, and $\mathrm{eig}(A) \cap \mathbb{C}^-$. Let $B(\lambda_0,\epsilon) \triangleq \{\lambda \in \mathbb{C} \mid \quad |\lambda - \lambda_0| < \epsilon\}$ denote the ϵ-ball around λ_0.

III. REVIEW

In this section we state some properties regarding fixed modes of a FDLTI system.

The following lemma will review the result regarding fixed modes with respect to a centralized sparsity pattern $\mathcal{S}_c$:

Lemma 3: For any plant P,

$$\Lambda(P,\mathcal{S}_c,\mathcal{T}^s) = \bigcup_{i=2,3,4} \mathrm{eig}\Big(\tilde{A}_{ii}\Big) \tag{7}$$

where $\tilde{A}_{ii}$ are the blocks in the Kalman (c.f. [8]) canonical decomposition of plant P, such that the fixed modes are the union of uncontrollable or unobservable modes of P.

Proof: See, for example, Theorem 2 in [9]. ∎

The following result regarding the fixed modes of an FDLTI system was established in the preceding paper [3]:

Theorem 4: For any FDLTI plant P, and any arbitrary information structure $\mathcal{S}$, the set of fixed modes w.r.t. static controllers in $\mathcal{S}$ is equivalent to the set of fixed modes w.r.t. dynamic controllers, i.e.:

$$\Lambda(P,\mathcal{S},\mathcal{T}^s) = \Lambda\big(P,\mathcal{S},\mathcal{T}^d\big). \tag{8}$$

Proof: See [3] for a comprehensive proof. ∎

In order to prove the main theorem, we will use the following lemma from [1]. The lemma states some properties regarding continuity and topology of non-fixed modes w.r.t. static controllers.

Lemma 5: For any plant P, and any information structure $\mathcal{S}$, partition the open-loop eigenvalues of P as in (5), then we have:

1) There exist $\epsilon > 0$, such that $B(\lambda,\epsilon) \subset \mathbb{C}^-$, for all $\lambda \in \Lambda_-(P)$.
2) For all $\epsilon > 0$, there exist $\gamma > 0$ such that for all $D \in \mathcal{S} \cap \mathcal{T}^s$ with $\|D\|_\infty < \gamma$, there are exactly $\mu(\lambda,P)$ eigenvalues of $A_{\mathrm{CL}}(P,D)$ in $B(\lambda,\epsilon)$, for all $\lambda \in \Lambda_+(P) \cup \Lambda_-(P)$.
3) For all $\gamma > 0$, there exist $D \in \mathcal{T}^s \cap \mathcal{S}$ with $\|D\|_\infty < \gamma$, such that $\lambda \notin \mathrm{eig}(A_{\mathrm{CL}}(P,D))$, for all $\lambda \in \Lambda_+(P) \cup \Lambda_-(P)$.

Proof: See, for example, Lemma 4 in [1]. The proof does not use any property specific to only block-diagonal information structure and thus could be replaced by any arbitrary information structure. ∎

Remark 6: It follows from the proof that non-admissible D that violate part 3 of Lemma 5, forms a subset with zero Lebesgue measure, thus a sufficiently small random $D \in \mathcal{S}$ satisfies all the conditions of Lemma 5. Precisely, the space of static controllers that does not satisfy part 3 of Lemma 5 construct a finite union of hyper-surfaces in $(\mathcal{T}^s \cap \mathcal{S}) \subset \mathbb{R}^{n_u \times n_y}$. Thus, a D that satisfies all the conditions of Lemma 5, can be found with probability one by randomly choosing direction of $D \in \mathcal{T}^s \cap \mathcal{S}$, and then scaling it appropriately such that $\|D\|_\infty < \gamma$.

IV. MAIN RESULT

With a constructive proof, we will show that we can stabilize a plant P with arbitrary information structure $\mathcal{S}$, as long as it has no unstable fixed modes. We will achieve this by showing that we can always find a controller which will reduce the number of unstable modes, while leaving all of the fixed modes in the LHP, which can then be applied as many times as required.

We first establish the following theorem.

Theorem 7: For any plant P with $|\Lambda_+(P)| \geq 1$, and fixed-modes in LHP (i.e., $\Lambda(P,\mathcal{S},\mathcal{T}^s) \subset \mathbb{C}^-$), there exist a $D_K \in \mathcal{T}^s \cap \mathcal{S}$, and an integer $m \in \{1,\cdots,a\}$, such that:

1) For some $\alpha \in \Lambda_+(P)$, α is a controllable and observable mode of the following SISO system:

$$\begin{aligned} P_m &= \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & 0 \end{array}\right] \\ &\triangleq \left[\begin{array}{c|c} A_P + B_P D_K C_P & B_P \mathbf{e}_{i_m} \\ \hline (\mathbf{e}_{j_m})^T C_P & 0 \end{array}\right] \end{aligned} \tag{9}$$

2) The total number of unstable modes of P_m is no greater than that of P, i.e.:

$$\nu(P_m) \leq \nu(P) \tag{10}$$

Proof: The proof is depicted as follows, we will first find a $D \in \mathcal{T}^s \cap \mathcal{S}$ that when closed around P, move all of its unstable eigenvalues, then based on this D, we will find a $D_m \in \mathcal{T}^s \cap \mathcal{S}$, such that at least one of the unstable modes of P, would also be an unstable mode of $\Gamma(P,D_{m-1})$ with multiplicity at least one, but there would be no common unstable mode between $\Gamma(P,D_m)$, and P, this means that only changing the $(i_m,j_m)^{\text{th}}$ element of D_{m-1}, will change the common unstable mode between P, and $\Gamma(P,D_{m-1})$, and thus those modes must be in the set of controllable and observable modes of the corresponding SISO plant from u_{i_m} to y_{j_m}.

Proof of part 1: partition eigenvalues of P as in (5), then based on Lemma 5.1, we can choose $\epsilon > 0$ small enough such that all the ϵ-balls around stable modes of P would be in LHP, this along with part 2, and 3 of Lemma 5, guarantees existence of a $D \in \mathcal{T}^s \cap \mathcal{S}$ such that due to Lemma 5.2, $A_{\mathrm{CL}}(P,D)$ has no greater number of unstable modes than that of P, $\left(\sum_{\alpha \in \Lambda_+(P)} \mu(\alpha,P)\right)$, and due to Lemma 5.3, $\alpha \notin \mathrm{eig}(A_{\mathrm{CL}}(P,D))$, for all $\alpha \in \Lambda_+(P)$.

Construct a sequence of matrices $D_m \in \mathcal{T}^s \cap \mathcal{S}$ as in (1), so that $D_a = D$, and $D_0 = 0$, thus:

$$\begin{aligned} \forall \alpha \in \Lambda_+(P): &\qquad \alpha \notin \mathrm{eig}(A_{\mathrm{CL}}(P,D_a)) \\ \forall \alpha \in \Lambda_+(P): &\qquad \alpha \in \mathrm{eig}(A_{\mathrm{CL}}(P,D_0)), \\ &\qquad \mu(\alpha,\Gamma(P,D_0)) = \mu(\alpha,P) \end{aligned}$$

By decreasing m from a to 1, there must exist a value of $m \in \{1,\cdots,a\}$, such that:

$$\forall \alpha \in \Lambda_+(P): \qquad \alpha \notin \mathrm{eig}(A_{\mathrm{CL}}(P,D_m)) \tag{11a}$$

$$\begin{aligned} \exists \alpha \in \Lambda_+(P): \qquad &\alpha \in \mathrm{eig}(A_{\mathrm{CL}}(P,D_{m-1})), \\ &\mu(\alpha,\Gamma(P,D_{m-1})) \geq 1 \end{aligned} \tag{11b}$$

thus if we set $D_K = D_{m-1}$ and use the definitions from (9), as illustrated in Figure 2, we have:

$$\begin{aligned}&\Lambda(\Gamma(P,D_m))\\ &=\operatorname{eig}(A_{\mathrm{CL}}(P,D_m))=\operatorname{eig}(A_P+B_P D_m C_P)\\ &=\operatorname{eig}\big(A_P+B_P D_{m-1}C_P+B_P\mathbf{e}_{i_m}D_{i_m,j_m}\mathbf{e}_{j_m}^T C_P\big)\\ &=\operatorname{eig}(A_m+B_m D_{i_m,j_m}C_m)=\operatorname{eig}(A_{\mathrm{CL}}(P_m,D_{i_m,j_m}))\end{aligned} \tag{12}$$

From (11b), there exists at least one $\alpha \in \Lambda_+(P)$ such that:

$$\begin{aligned}\alpha \in \operatorname{eig}(A_{\mathrm{CL}}(P,D_{m-1})) \quad &= \quad \operatorname{eig}(A_m),\\ &\mu(\alpha,\Gamma(P,D_{m-1}))\geq 1\end{aligned}$$

but due to (11a),

$$\alpha \notin \operatorname{eig}(A_{\mathrm{CL}}(P,D_m)) \overset{(12)}{=} \operatorname{eig}(A_{\mathrm{CL}}(P_m,D_{i_m,j_m}))$$

hence, for all such α that are moved by only closing D_{i_m,j_m} around the SISO system P_m (for which the only information structure is the centralized one, $\mathcal{S}_{\mathrm{c}}$), we have:

$$\begin{aligned}\exists\, D_{i_m,j_m}\in\mathbb{R} \quad \text{s.t.:} \quad &\alpha\notin\operatorname{eig}(A_{\mathrm{CL}}(P_m,D_{i_m,j_m}))\\ &\Rightarrow \alpha\notin\Lambda(P_m,\mathcal{S}_{\mathrm{c}},\mathcal{T}^{\mathrm{s}})\end{aligned}$$

and finally, since, due to Lemma 3, fixed modes of any FDLTI plant with centralized information structure is equal to its unobservable or uncontrollable modes, we must have that those α are controllable and observable mode of P_m.

Proof of part 2: since ϵ-balls in Lemma 5.1 are chosen sufficiently small to keep the stable modes in LHP, then the resulting D that is described in proof of part 1 is such that $A_{\mathrm{CL}}(P,D)=A_P+B_P D C_P$ has no more unstable eigenvalue than P itself (due to Lemma 5.2), also due to definition of matrices D_m, for each $m\in\{0,1,\cdots,a\}$, $\|D_m\|_\infty \leq \|D\|_\infty \leq \gamma$, and thus from Lemma 5.2, $A_m = A_P + B_P D_{m-1} C_P$ has no more unstable eigenvalue (counting with their multiplicities) than that of P. ■

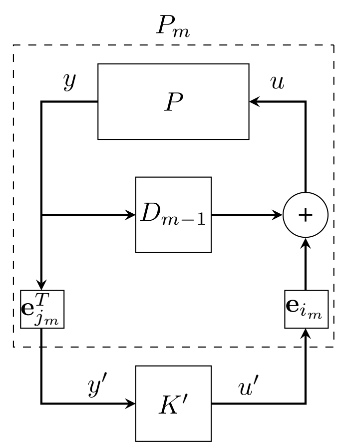


Fig. 2: P_m is the SISO map from u' to y', and K_m is the map from y to u, giving the total control for the original plant.

In the following Proposition, we will take advantage of the well known result regarding arbitrarily placing controllable and observable modes of a plant with no information constraint on the controller by designing a observer-based controller to stabilize some of the unstable modes of P_m, defined in (9), more specifically those that are controllable and observable. We will add one further design constraint that unstable modes of controller would be different than that of P_m, and will show that this constraint is always feasible. This ensures that an induction argument can be used later on. This constraint is not mentioned in [1] even for diagonal information structure, and it is unclear that without such a constraint how the induction could follow, even for the diagonal information structure.

Proposition 8: All the controllable and observable unstable modes of the plant P_m can be stabilized by an observer-based controller K' such that:

$$\Lambda_+(K') \cap \Lambda_+(\Gamma(P_m,K')) = \varnothing. \tag{13}$$

Proof: Our proof is in a constructive manner, we will first find a K' to only stabilize the controllable and observable modes of P_m without considering (13), and then do an appropriate perturbation when it is not met by K' at the end.

First find a similarity transformation T that will put P_m in its Kalman canonical form, therefore we would have:

$$\begin{bmatrix}T & 0\\ 0 & I\end{bmatrix}\left[\begin{array}{c|c}A_m & B_m\\ \hline C_m & 0\end{array}\right]\begin{bmatrix}T^{-1} & 0\\ 0 & I\end{bmatrix}=$$
$$\left[\begin{array}{cccc|c}\tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 & \tilde{B}_1\\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{B}_2\\ 0 & 0 & \tilde{A}_{33} & 0 & 0\\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} & 0\\ \hline \tilde{C}_1 & 0 & \tilde{C}_2 & 0 & 0\end{array}\right] \tag{14}$$

it is notable that all the $\tilde{(\cdot)}$ parameters depend on the transformation matrix T and the state-space representation of P_m. We want to stabilize all the unstable modes in $\tilde{A}_{11}$. Since based on definition $(\tilde{A}_{11},\tilde{B}_1)$ is a controllable pair and $(\tilde{A}_{11},\tilde{C}_1)$ is an observable pair, there exists a state feedback gain F, and an observer gain L, such that eigenvalues of $\tilde{A}_{11}-\tilde{B}_1 F$, and $\tilde{A}_{11}-L\tilde{C}_1$ can be arbitrarily assigned, and hence can be stabilized. Take:

$$K'=\left[\begin{array}{c|c}A' & B'\\ \hline C' & 0\end{array}\right]=\left[\begin{array}{c|c}\tilde{A}_{11}-\tilde{B}_1F-L\tilde{C}_1 & L\\ \hline -F & 0\end{array}\right]. \tag{15}$$

With the help of two similarity transformations, we have:

$$\begin{aligned}&\operatorname{eig}(A_{\mathrm{CL}}(P_m,K'))=\\ &\operatorname{eig}\left(T_1 A_{\mathrm{CL}}(\begin{bmatrix}T & 0\\ 0 & I\end{bmatrix}P_m\begin{bmatrix}T^{-1} & 0\\ 0 & I\end{bmatrix},K')T_1^{-1}\right)=\operatorname{eig}\end{aligned}$$
$$\left(\begin{array}{cc|ccc}\tilde{A}_{11}-\tilde{B}_1F & 0 & \tilde{A}_{13} & 0 & -\tilde{B}_1F\\ \tilde{A}_{21}-\tilde{B}_2F & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & -\tilde{B}_2F\\ \hline 0 & 0 & \tilde{A}_{33} & 0 & 0\\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} & 0\\ 0 & 0 & L\tilde{C}_2-\tilde{A}_{13} & 0 & \tilde{A}_{11}-L\tilde{C}_1\end{array}\right), \tag{16}$$

with T being the same transformation which puts P_m in its Kalman canonical form, and T_1 keeps the first four rows the same and subtract first row from fifth. Thus the eigenvalue of the closed loop would be

$$\begin{aligned}&\operatorname{eig}(A_{\mathrm{CL}}(P_m,K'))=\\ &\operatorname{eig}\Big(\tilde{A}_{11}-\tilde{B}_1F\Big)\cup\operatorname{eig}\Big(\tilde{A}_{11}-L\tilde{C}_1\Big)\cup\left(\bigcup_{i=2}^{4}\operatorname{eig}\Big(\tilde{A}_{ii}\Big)\right)\end{aligned} \tag{17}$$

therefore for all observer-based controllers that naturally satisfy $\text{eig}\left(\tilde{A}_{11}-\tilde{B}_1 F\right)\in\mathbb{C}^-$, and $\text{eig}\left(\tilde{A}_{11}-L\tilde{C}_1\right)\in\mathbb{C}^-$; unstable modes of $\Gamma(P_m,K')$ would be independent of F, and L. Thus we have:

$$\Lambda_+(\Gamma(P_m,K'))=\bigcup_{i=2}^{4}\Lambda_+(\tilde{A}_{ii}), \tag{18}$$

and all unstable modes in $\tilde{A}_{11}$ can be stabilized by appropriate choice of matrices L, and F. According to (18), constraint (13) is not met if and only if

$$\Lambda_+(K')\bigcap\left(\bigcup_{i=2}^{4}\Lambda_+(\tilde{A}_{ii})\right)\neq\varnothing, \tag{19}$$

in this case, we enforce (13) by appropriately perturbing the L matrix. We claim that there exist a perturbation $\hat{L}$ such that for $\hat{K}'$ defined as:

$$\hat{K}'\triangleq\left[\begin{array}{c|c}\tilde{A}_{11}-\tilde{B}_1F-(L+\hat{L})\tilde{C}_1 & L+\hat{L}\\ \hline -F & 0\end{array}\right] \tag{20}$$

then (13) is satisfied. To see this first define W as:

$$W\triangleq\left[\begin{array}{c|c}\tilde{A}_{11}-\tilde{B}_1F-L\tilde{C}_1 & I\\ \hline \tilde{C}_1 & 0\end{array}\right]. \tag{21}$$

If we close static L around W, we have:

$$A_{\text{CL}}(W,L)=\tilde{A}_{11}-\tilde{B}_1F-L\tilde{C}_1+L\tilde{C}_1=\tilde{A}_{11}-\tilde{B}_1F,$$

and thus we must have that $\Lambda(W,\mathcal{S}_\text{c},\mathcal{T}^\text{s})\subset\text{eig}\left(\tilde{A}_{11}-\tilde{B}_1F\right)$. Since F is chosen to stabilize $\tilde{A}_{11}$, we have $\text{eig}\left(\tilde{A}_{11}-\tilde{B}_1F\right)\subset\mathbb{C}^-$, hence $\Lambda\,(W,\mathcal{S}_\text{c},\mathcal{T}^\text{s})\subset\mathbb{C}^-$. Also since L is chosen such that $\text{eig}\left(\tilde{A}_{11}-L\tilde{C}_1\right)\subset\mathbb{C}^-$, and thus due to a continuity argument, for all sufficiently small $\hat{L}$, $\text{eig}\left(\tilde{A}_{11}-(L+\hat{L})\tilde{C}_1\right)\subset\mathbb{C}^-$, i.e.:

$$\exists\,\gamma>0\colon\ \text{eig}\left(\tilde{A}_{11}-(L+\hat{L})\tilde{C}_1\right)\subset\mathbb{C}^-\quad\forall\,\hat{L}\ \text{with}\ \|\hat{L}\|_\infty<\gamma.$$

Applying Lemma 5.3 with $P=W$, and $\mathcal{S}=\mathcal{S}_\text{c}$, guarantees existence of a perturbation $\hat{L}$, such that all the non fixed-modes of W (which include all the $\Lambda_+(W)$) would be changed by closing $\hat{L}$ around W, i.e.:

$$\exists\,\hat{L}\quad\text{with}\quad\|\hat{L}\|_\infty<\gamma\quad\text{s.t.:}$$

$$1.\qquad \text{eig}\left(\tilde{A}_{11}-(L+\hat{L})\tilde{C}_1\right)\subset\mathbb{C}^-$$

$$2.\qquad \Lambda_+(\Gamma(W,\hat{L}))\bigcap\left(\bigcup_{i=2}^{4}\Lambda_+(\tilde{A}_{ii})\right)=\varnothing.$$

Since $\Lambda_+(\Gamma(W,\hat{L}))=\Lambda_+(\tilde{A}_{11}-\tilde{B}_1F-(L+\hat{L})\tilde{C}_1)$, by perturbing L by $\hat{L}$, all the conditions of Proposition 8 would be met if we use $\hat{K}'$ instead of K'. ■

Corollary 9: For every plant P that satisfies the assumptions of Theorem 7, there exists a controller K_m, such that for some $m\in\{1,\cdots,a\}$, $K_m\in\mathcal{T}^{\text{s}+1}_{i_m,j_m}\cap\mathcal{S}$, and:

$$A_{\text{CL}}(P_m,K') = A_{\text{CL}}(P,K_m), \tag{22}$$

$$\nu(\Gamma(P,K_m)) \leq \nu(P)-1, \tag{23}$$

$$\Lambda_+(K_m)\cap\Lambda_+(\Gamma(P,K_m)) = \varnothing, \tag{24}$$

where P_m and K' are determined as above.

Proof: Use Theorem 7 to find D_K, and m, and Proposition 8 to find K', and construct the MIMO controller $K_m=D_{m-1}+\mathbf{e}_{i_m}K'\mathbf{e}_{j_m}^T$. As illustrated in Figure 2, this K_m that has the following state-space representation

$$K_m=\left[\begin{array}{c|c}A_m^K & B_m^K\\ \hline C_m^K & D_m^K\end{array}\right]=\left[\begin{array}{c|c}A' & B'\mathbf{e}_{j_m}^T\\ \hline \mathbf{e}_{i_m}C' & D_K\end{array}\right] \tag{25}$$

satisfies (22). Due to Theorem 7 and Proposition 8, K' will stabilize at least one unstable mode of P, thus we have $\nu(\Gamma(P_m,K'))\leq\nu(P)\ -1$, then (23) would be an immediate result of this property of K' combined with (22). Furthermore, as per construction (25), we have $A_m^K=A'$, thus $\Lambda_+(K')=\Lambda_+(K_m)$, similarly, because of (22), we have $\Lambda_+(\Gamma(P_m,K'))=\Lambda_+(\Gamma(P,K_m))$, thus (24) follows from (13). ■

We use induction to prove that if all the fixed modes of P are in LHP, then we can stabilize P by dynamic controller. We will first define the following interconnection that will be useful in the induction. Let $G^{(0)}\triangleq P$ and at each step k, denote the transfer function from u to y, as illustrated in Figure 3, by $G^{(k+1)}$, i.e., $G^{(k+1)}=\Gamma(G^{(k)},K_m^{(k)})$. Let $(A_G^{(k)},B_G^{(k)},C_G^{(k)},0)$ be a state-space representation for $G^{(k)}$, also denote the total number of unstable modes of $G^{(k)}$ by $\nu^{(k)}\triangleq\sum_{\alpha\in\Lambda_+(G^{(k)})}\mu(\alpha,G^{(k)})$.

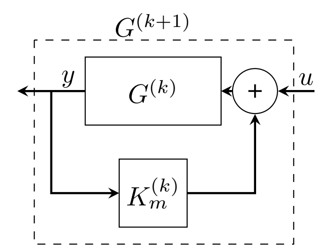


Fig. 3: Plant $G^{(k+1)}\triangleq\Gamma(G^{(k)},K_m^{(k)})$.

The induction is such that in each step k, we will find an integer $m^{(k)}\in\{1,\cdots,a\}$, and a $K_m^{(k)}\in\mathcal{T}^{\text{s}+1}_{i_{m^{(k)}},j_{m^{(k)}}}\cap\mathcal{S}$ that when closed around $G^{(k)}$, will stabilize at least one unstable mode of $G^{(k)}$, thus $\nu^{(k+1)}\leq\nu^{(k)}-1$. Then we will treat the corresponding $G^{(k+1)}$ as the new plant for which we want to stabilize the rest of remaining $\nu^{(k+1)}$ unstable eigenvalues. Thus in at most $\nu^{(0)}$ steps, P will be stabilized. A crucial part of induction is that $G^{(k+1)}$ must have no fixed-mode in closed RHP, this is not addressed in [1] and at this point it is directly claimed that Theorem 12 holds true. We will formalize this fact with the help of following lemma. It is enough to show that closing K_m around P does not add any unstable fixed modes to $\Gamma(P,K_m)$.

Lemma 10: Assume that all the fixed modes of P are in LHP, i.e.:

$$\Lambda(P,\mathcal{S},\mathcal{T}^\text{s})\subset\mathbb{C}^-, \tag{26}$$

also, assume that a controller K_m is such that it satisfies (24), then we have:

$$\Lambda(\Gamma(P,K_m),\mathcal{S},\mathcal{T}^\text{s})\subset\mathbb{C}^-. \tag{27}$$

Proof: Proof is done by contradiction, we will first create the following set-up to state the idea. Let (A_K,B_K,C_K,D_K)

be a state-space representation for K_m. We have:

$$\Lambda(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}) \subseteq \Lambda(\Gamma(P,K_m),\mathcal{S},\mathcal{T}^{\mathrm{s}}), \tag{28}$$

since otherwise, if $\lambda \notin \Lambda(\Gamma(P,K_m),\mathcal{S},\mathcal{T}^{\mathrm{s}})$, then, by Definition 1, there exist a K_2, such that, $\lambda \notin \Gamma(\Gamma(P,K_m), K_2) \overset{(3)}{=} \Gamma(P,\ K_m{+}K_2)$, thus $\lambda \notin \Lambda\left(P,\mathcal{S},\mathcal{T}^{\mathrm{d}}\right) \overset{(8)}{=} \Lambda\left(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}\right)$. Next, It is trivial to check that if we close $-K_m$ around $\Gamma(P,K_m)$, then by applying a similarity transformation T_2, $\Gamma(\Gamma(P,K_m),-K_m)$ can be written as:

$$\begin{bmatrix} T_2 & 0 \\ 0 & I \end{bmatrix} \Gamma(\Gamma(P,K_m),-K_m) \begin{bmatrix} T_2^{-1} & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{cc:c|c} A_P & B_P C_K & 0 & B_P \\ 0 & A_K & 0 & 0 \\ \hdashline B_K C_P & 0 & A_K & 0 \\ \hline C_P & 0 & 0 & 0 \end{array}\right], \tag{29}$$

thus we have

$$\mathrm{eig}(\Gamma(\Gamma(P,K_m),-K_m)) = \mathrm{eig}(A_P) \cup \mathrm{eig}(A_K).$$

Furthermore, due to (26), there exist a $D \in \mathcal{T}^{\mathrm{s}} \cap \mathcal{S}$ that will move all the unstable modes of A_P, we can apply the same D on (29) to move all non fixed-modes of A_P, thus we have:

$$\Lambda(\Gamma(P,K_m),\mathcal{S},\mathcal{T}^{\mathrm{s}}) \subseteq \Lambda(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}) \cup \mathrm{eig}(A_K). \tag{30}$$

Now we are ready to do the main contradiction part, assume that there exist an $\alpha \in \Lambda\left(\Gamma(P,K_m),\mathcal{S},\mathcal{T}^{\mathrm{s}}\right)$, with $\Re(\alpha) \geq 0$, then

$$\begin{aligned} &\alpha \in \Lambda(\Gamma(P,K_m),\mathcal{S},\mathcal{T}^{\mathrm{s}}), && \Re(\alpha) \geq 0 \\ &\alpha \overset{(30)}{\in} \Lambda(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}) \cup \mathrm{eig}(A_K) && \overset{(26)}{\Rightarrow} \alpha \in \mathrm{eig}(A_K) \\ &\overset{(24)}{\Rightarrow} \alpha \notin \mathrm{eig}(\Gamma(P,K_m)) && \Rightarrow \alpha \notin \Lambda(\Gamma(P,K_m),\mathcal{S},\mathcal{T}^{\mathrm{s}}), \end{aligned}$$

thus we have achieved the desired contradiction. ∎

In Lemma 10, constraint (24) is a sufficient condition to make the proof achievable and is always feasible if we want to only stabilize the plant P, however, since the perturbation $\hat{L}$ changes modes of $\tilde{A}_{11} - (L + \hat{L})\tilde{C}_1$, this further design constraint may prevent specific pole assignment when it is not met. Nevertheless, by choosing $\hat{L}$ sufficiently small, we can place poles arbitrary close to the desired positions.

Now we are ready to claim that if all the fixed modes of P are in LHP, then we can stabilize P by a dynamic controller. This stabilizing controller would be a summation of individual controllers $K_m^{(k)}$, each obtained in one step of the induction, where in each step k, $K_m^{(k)}$ would only have one dynamic element (i.e., $K_m^{(k)} \in \mathcal{T}^{\mathrm{s}+1}_{i_{m^{(k)}},j_{m^{(k)}}} \cap \mathcal{S}$, for some $m^{(k)} \in \{1,\cdots,a\}$).

Theorem 11: For any plant P, and any information structure $\mathcal{S}$, if $\Lambda(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}) \subset \mathbb{C}^-$, then there exist a stabilizing controller $K \in \mathcal{T}^{\mathrm{d}} \cap \mathcal{S}$ that will stabilize P.

Proof: Proof is done by induction. Let $G^{(0)} \triangleq P$. As per assumption of this theorem, $\Lambda\left(G^{(0)},\mathcal{S},\mathcal{T}^{\mathrm{s}}\right) = \Lambda(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}) \subset \mathbb{C}^-$. At each step k, Corollary 9 suggests a method, by which we can find a controller that will stabilize at least one of unstable modes of $G^{(k)}$. Specifically, with P replaced by $G^{(k)}$ in Corollary 9, we can find a $m^{(k)} \in \{1,\cdots,a\}$, and a controller $K_m^{(k)} \in \mathcal{T}^{\mathrm{s}+1}_{i_{m^{(0)}},j_{m^{(0)}}} \cap \mathcal{S}$, that will stabilize at least one of unstable modes of $G^{(k)}$, thus:

$$\nu(G^{(k+1)}) = \nu(\Gamma(G^{(k)},K_m^{(k)})) \leq \nu(G^{(k)}) - 1.$$

This $K_m^{(k)}$ satisfies (24) (with P replaced by $G^{(k)}$), and thus by Lemma 10, $G^{(k+1)} = \Gamma(G^{(k)},K_m^{(k)})$, would have all of its fixed modes in LHP, i.e., $\Lambda\left(G^{(k+1)},\mathcal{S},\mathcal{T}^{\mathrm{s}}\right) \in \mathbb{C}^-$. This guarantees that we can use Corollary 9 on $G^{(k+1)}$, if it has at least one unstable mode. Since at each step, at least one unstable mode is stabilized, P could be stabilized in at most $\nu(P)$ steps. The final $K \in \mathcal{T}^{\mathrm{d}} \cap \mathcal{S}$ that will stabilize P, is equal to the summation of controllers at each step, i.e.:

$$K(\sigma) \overset{(3)}{=} \sum\nolimits_k K_m^{(k)}(\sigma). \tag{31}$$

∎

We can easily show that stability of all the fixed-modes of P, $\Lambda(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}) \subset \mathbb{C}^-$, is also a necessary condition for the existence of stabilizing controller, which is formalized in the following theorem:

Theorem 12: A plant P is stabilizable by a controller $K \in \mathcal{T}^{\mathrm{d}} \cap \mathcal{S}$, if and only if $\Lambda(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}) \subset \mathbb{C}^-$.

Proof: The sufficiency part is done in Theorem 11. For the necessity part note that by definition, fixed-modes of P can not be moved by admissible static controllers, and due to Theorem 4, they can not be moved by admissible dynamic controllers either. Hence, if a stabilizing controller is found for P, we must have $\Lambda(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}) \subset \mathbb{C}^-$, i.e.:

$$\begin{aligned} &\Lambda(P,\mathcal{S},\mathcal{T}^{\mathrm{s}}) \not\subset \mathbb{C}^- \quad \overset{\text{Thm.4}}{\Rightarrow} \Lambda\left(P,\mathcal{S},\mathcal{T}^{\mathrm{d}}\right) \not\subset \mathbb{C}^- \\ &\overset{\text{bydef}}{\Rightarrow} \nexists K \in \mathcal{T}^{\mathrm{d}} \cap \mathcal{S} \qquad \text{s.t.} \qquad A_{\mathrm{CL}}(P,K) \subset \mathbb{C}^-. \end{aligned}$$

∎

V. Synthesis and Numerical Example

In this section we provide an explicit algorithm to stabilize a plant which has no unstable fixed modes, and run it on one numerical example to illustrate its implementation. Algorithm 1 is distilled from the steps taken in the paper to prove the main theorem, and thus can almost certainly be improved upon in several respects.

In Algorithm 1, $D^{(k)}$ is randomly chosen, and as stated in remark 6, is a valid choice with probability one. This $D^{(k)}$ must be chosen small enough ($\|D^{(k)}\|_\infty < \gamma^{(k)}$) such that the total number of unstable modes would not increase when each element of the sequence $\{D_m^{(k)}\}_{m=1}^a$ is closed around $G^{(k)}$. A prior knowledge of such an upper bound on $D^{(k)}$ ($\gamma^{(k)}$) is not available and is hard to attain. This we consider making $D^{(k)}$ small enough in a **repeat - until** loop such that the statement in proof of Theorem 7.2 holds true. This iterative scaling repeats itself when (13) is not met. In that case, perturbation $\hat{L}^{(k)}$ is chosen small enough (by a similar loop) so that it will not make any modes of $\tilde{A}_{11}^{(k)} - (L^{(k)} + \hat{L}^{(k)})\tilde{C}_1^{(k)}$ unstable.

The following numerical example will use Algorithm 1 to stabilize the plant P.

Example 13: Consider the following plant:

$$A = \mathrm{diag}(2,3,5,-1,-1)$$

$$B = \begin{bmatrix} 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 4 & 0 & 8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}, D = 0.$$

Algorithm 1 Finding a controller $K \in \mathcal{T}^{\mathrm{d}} \cap \mathcal{S}$ to stabilize P

Input: Plant P, information structure $\mathcal{S}$
Output: Controller $K \in \mathcal{T}^{\mathrm{d}} \cap \mathcal{S}$ that will stabilize P
  $k \leftarrow 0,\ G^{(0)} \leftarrow P,\ K(\sigma) \leftarrow 0$
  while $|\Lambda_+(G^{(k)})| \geq 1$ **do**
    choose a random $D^{(k)} \in \mathcal{T}^{\mathrm{s}} \cap \mathcal{S}$
    $D^{(k)} \leftarrow 2D^{(k)}$
    repeat
      $D^{(k)} \leftarrow D^{(k)}/2$
      $m^{(k)} \leftarrow a$
      while $\Lambda_+(\Gamma(G^{(k)}, D^{(k)}_{m^{(k)}-1})) \cap \Lambda_+(G^{(k)}) = \varnothing$ **do**
        $m^{(k)} \leftarrow m^{(k)} - 1$
      end while
    until $\max\limits_{j=m^{(k)}-1, m^{(k)}} \nu(\Gamma(G^{(k)}, D^{(k)}_j)) \leq \nu(G^{(k)})$
    $G^{(k)}_{m^{(k)}} \leftarrow \mathbf{e}^T_{j_{m^{(k)}}} \Gamma(G^{(k)}, D^{(k)}_{m^{(k)}-1}) \mathbf{e}_{i_{m^{(k)}}}$
    find a Kalman similarity transformation $T^{(k)}$ for $G^{(k)}_{m^{(k)}}$
    name all the corresponding partitions by $\tilde{(\cdot)}^{(k)}$
    find a $F^{(k)}$ to stabilize $\tilde{A}^{(k)}_{11} - \tilde{B}^{(k)}_1 F^{(k)}$
    find a $L^{(k)}$ to stabilize $\tilde{A}^{(k)}_{11} - L^{(k)} \tilde{C}^{(k)}_1$
    if $\Lambda_+(\tilde{A}^{(k)}_{11} - \tilde{B}^{(k)}_1 F^{(k)} - L^{(k)} \tilde{C}^{(k)}_1) \cap \left(\bigcup_{i=2}^{4} \Lambda_+(\tilde{A}^{(k)}_{ii}) \right) \neq \varnothing$ **then**
      choose a random $\hat{L}^{(k)}$
      while $|\Lambda_+\left(\tilde{A}^{(k)}_{11} - (L^{(k)} + \hat{L}^{(k)}) \tilde{C}^{(k)}_1\right)| \geq 1$ **do**
        $\hat{L}^{(k)} \leftarrow \hat{L}^{(k)}/2$
      end while
      $L^{(k)} \leftarrow L^{(k)} + \hat{L}^{(k)}$
    end if
    $K^{(k)} \leftarrow \left[\begin{array}{c|c} \tilde{A}^{(k)}_{11} - \tilde{B}^{(k)}_1 F^{(k)} - L^{(k)} \tilde{C}^{(k)}_1 & L^{(k)} \mathbf{e}^T_{j_{m^{(k)}}} \\ \hline -\mathbf{e}_{i_{m^{(k)}}} F^{(k)} & D^{(k)}_{m^{(k)}-1} \end{array}\right]$
    $K(\sigma) \leftarrow K(\sigma) + K^{(k)}(\sigma)$
    $G^{(k+1)} \leftarrow \Gamma(G^{(k)}, K^{(k)})$
    $k \leftarrow k + 1$
  end while
  return $K(\sigma)$

Let controller information constraint be given by: $\mathrm{Adm}(\mathcal{S}) = \{(1,1),(3,1),(4,1),(5,2),(1,3),(3,3),(4,3),(5,4),(5,5)\}$. This plant has fixed mode $\Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{s}}\right) = \{-1\}$. If we follow Algorithm 1 to stabilize P, then closed-loop modes $\begin{bmatrix} -0.5 & -1 & -1 & -1.5 & -2 & -2.5 & -3 & -3.5 \end{bmatrix}^T$ are achieved by the following controller:

$$\left[\begin{array}{ccc|ccc} 14.92 & -460.41 & -4.66 & & 317.12 & \\ 0.37 & -24.44 & 0.75 & 0_{3\times 1} & 27.44 & 0_{3\times 3} \\ 22.92 & -763.84 & -25.42 & & 405.62 & \\ \hline & 0_{1\times 3} & & 0.09 & & \\ & 0_{1\times 3} & & 0 & & \\ & 0_{1\times 3} & & 0.43 & 0_{5\times 1} & 0_{5\times 3} \\ & 0_{1\times 3} & & 0.09 & & \\ 3.90 & -71.64 & -7.45 & 0 & & \end{array}\right]$$

An alternative approach is taken in [2], in which, at each step, a (possibly dynamic) stabilizing controller is applied at the next diagonal element of the controller, and it is shown that by adding stabilizing controllers at each step, the set of (possibly unstable) fixed modes are reduced, until the last step where the remaining fixed modes must be necessarily stable. Applying method of [2] on this plant, would result in a stabilizing controller of order 7, as compared to 3 here. An explanation could be that in [2], a (possibly dynamic) stabilizing controller is applied at each of the elements, resulting in abundant of controller states, whereas in here, only for each unstable mode, a stabilizing controller (not necessarily of order 1) is needed.

If we look at each of the nine SISO maps $\{P_{i_m j_m}\}_{m=1}^{9}$, then the union of controllable and observable modes of all these SISO maps are $\{2, 5\}$, which does not contain the unstable mode 3. This shows that if we follow Algorithm 1, then a static gain (the D_{m-1} of Figure 2) may be necessary before an observer-based controller can be used to stabilize that mode. This is counter-intuitive compared to the centralized case where a stabilizing observer-based controller would have zero static gain.

VI. Conclusion

We revisited, verified, and generalized classic work in the stabilizability of decentralized systems. In a preceding paper, we generalized the notion of fixed modes to arbitrary information structure, and provided a rigorous inductive proof that plant modes which cannot be moved by static LTI controllers with the prescribed structure cannot be moved by dynamic LTI controllers either. In this paper, we addressed the placement of the modes which are not fixed. We showed that they can be moved to within a chosen accuracy of any desired pole locations, thus similarly solidifying and generalizing the other main result of [1]. Combining these results, we have shown that having fixed modes in the LHP w.r.t. static LTI controllers of a given information structure is necessary and sufficient for stabilizability by dynamic LTI controllers with the same structure. We lastly presented an explicit algorithm for finding such a stabilizing decentralized controller.

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