Parameterization of Causal Stabilizing Controllers over Networks with Delays

Michael Rotkowitz[1,2]

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Abstract

We consider the problem of multiple subsystems, each with its own controller, such that the dynamics of each subsystem may affect those of other subsystems with some propagation delays, and the controllers may communicate with each other with some transmission delays. It was recently shown, assuming linearity and time-invariance of the subsystems and their controllers, that if the transmission delays satisfy the triangle inequality, then the simple condition that the transmission delay between any two subsystems is less than the propagation delay between those subsystems allows for the optimal control problem to be recast as a convex optimization problem. In this paper it is shown that the same condition allows for parameterization of all causal stabilizing decentralized controllers, even if the subsystems or admissible controllers are nonlinear time-varying.

1 Introduction

We consider the problem of multiple subsystems, each with its own controller, such that the dynamics of each subsystem may effect those of other subsystems with some propagation delay, and the controllers may communicate with each other with some transmission delays. We seek to parameterize all of the causal controllers with these delays which stabilize the system. This paper states simple conditions on the delays such that this parameterization can be achieved.

It has been shown for general constrained control that a particular property, which in this paper we refer to as (1)-invariance, allows for a parameterization of all causal stabilizing controllers which satisfy the desired constraint [5]. We thus achieve our characterization of delays which allow for parameterization by testing for (1)-invariance.

We find that if the transmission delays satisfy the triangle inequality, and if the propagation delay between any pair of subsystems is at least as large as the transmission delay between those subsystems, then the problem is (1)-invariant. In other words, if data can be transmitted faster than dynamics propagate along any link, then all causal stabilizing controllers may be parameterized.

It is important to note the extreme generality of this framework and of this result. It holds for discrete-time systems and continuous-time systems. It does not assume that either the dynamics of the subsystems nor their controllers are either linear or time-invariant. It does not assume that the dynamics of any subsystem are the same as those of any other, and they may all be completely different types of objects. Most importantly, the delay between any two subsystems is not assumed to have any relationship whatsoever to other delays in the system. They may be assigned independently for each link.

Prior Work. A vast amount of prior work on optimal control over networks assumes that the actions of any subsystem have no effect on the dynamics of other subsystems. For a few other specific structures, tractable methods have been found [9, 8, 2, 4, 7]. A brief history of these may be found in [6].

In [6], it was shown that for an arbitrary network, if the propagation delay along any link exceeds the transmission delay along that link, then the set of stabilizing decentralized controllers can be parameterized, and the optimal decentralized control problem can be cast as a convex optimization problem. This unified and generalized these other results, and it assumed that the subsystems and their possible controllers were all linear time-invariant. In this paper, it is shown that the same simple conditions allow for parameterization of all causal stabilizing decentralized controllers, where both the subsystems and their controllers may be nonlinear time-varying.

Outline. In Section 2, we state some preliminaries and notation, define the propagation and transmission delays, explain why we may assume that the transmission delays satisfy the triangle inequality, formulate the problem we wish to solve, define (1)-invariance, and give an overview of its implications, in particular, that it allows parameterization of all stabilizing causal admissible controllers.

Section 3 contains the main result of the paper, where we prove that if this triangle inequality is satisfied, and if

[1] Research School of Information Sciences and Engineering, The Australian National University, Canberra, Australia, (e-mail: rotkowitz@stanfordalumni.org).

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the propagation delay associated with any pair of subsystems is at least as large as the associated transmission delay, then the information constraint is (1)-invariant, and thus, we can parameterize all stabilizing causal controllers.

In Section 3.1 we break these total transmission delays out into a pure transmission delay, representing the time it takes to communicate the information from one subsystem to another, and a computational delay, representing the time it takes to process the information before it is used by the controller. We find, somewhat surprisingly, that transmitting faster than the propagation of dynamics still allows for this parameterization, and in fact, that the computational delay causes the condition to be relaxed.

We make some concluding remarks in Section 4.

2 Preliminaries

All of the statements in this paper can be made both for continuous time and for discrete time. Rather than state each of these twice, we use $\mathcal{T}_+$ to refer to both $\mathbb{R}_+$ and $\mathbb{Z}_+$, and then use $\mathcal{L}_e^{m\times n}$ to refer to the corresponding extended space, which can then refer either to $L_{pe}^{m\times n}$ or $\ell_e^{m\times n}$.

When the dimensions are implied by context, we omit the superscripts of $\mathcal{L}_e^{m\times n}$.

We define the truncation operator P_T for all $T \in \mathcal{T}_+$ on all functions $f : \mathcal{T}_+ \to \mathbb{R}$ such that $f_T = P_T f$ is given by

$$f_T(t) = \begin{cases} f(t) & \text{if } t \leq T \\ 0 & \text{if } t > T \end{cases}$$

An operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is said to be causal iff

$$P_T H P_T \;=\; P_T H \qquad \forall\, T \in \mathcal{T}_+$$

that is, iff future inputs can't affect past or present outputs.

In the remainder of this paper, we assume that the technical conditions of [5] hold.

Stability. A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is said to be finite gain stable, hereafter abbreviated as f.g. stable, iff there exists $\gamma < \infty$ such that $\forall\, T \in \mathcal{T}_+$ and $\forall\, x \in \mathcal{L}_e$

$$\|P_T H x\| \;\leq\; \gamma \|P_T x\|$$

A causal operator $H : \mathcal{L}_e \to \mathcal{L}_e$ is said to be incrementally stable, hereafter abbreviated as inc. stable, iff there exists $\gamma < \infty$ such that $\forall\, T \in \mathcal{T}_+$ and $\forall\, x, y \in \mathcal{L}_e$

$$\|P_T H x - P_T H y\| \;\leq\; \gamma \|P_T x - P_T y\|$$

We say that $K : \mathcal{L}_e \to \mathcal{L}_e$ f.g. stabilizes $G : \mathcal{L}_e \to \mathcal{L}_e$ iff for the interconnection in Figure 1 the maps from the two inputs to the four other signals are all f.g. stable.

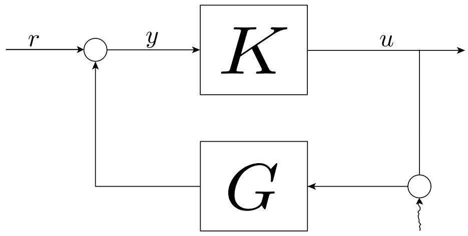

Figure 1: Interconnection of K and G

We say that $G : \mathcal{L}_e \to \mathcal{L}_e$ is strongly stabilizable iff there exists $K : \mathcal{L}_e \to \mathcal{L}_e$ which is inc. stable and which f.g. stabilizes G.

Delay. We define Delay$(\cdot)$ for a causal operator as the smallest amount of time in which an it can affect its output. For any causal $H : \mathcal{L}_e^m \to \mathcal{L}_e^n$,

$$\begin{aligned} \text{Delay}(H) \;=\; \inf\{\tau \geq 0 \mid\; & z_1(T+\tau) \neq z_2(T+\tau), \\ & z_1 = H(w_1), z_2 = H(w_2), \\ & w_1, w_2 \in \mathcal{L}_e^m, \\ & w_1(t) = w_2(t)\ \forall\ t \leq T\} \end{aligned}$$

and if $H = 0$, we consider its delay to be infinite.

Note that we then have the following inequalities for the delays of a composition or an addition of operators:

$$\begin{aligned} \text{Delay}(AB) &\geq \text{Delay}(A) + \text{Delay}(B) \\ \text{Delay}(A+B) &\geq \min\{\text{Delay}(A), \text{Delay}(B)\} \end{aligned}$$

We will need one more relation due to the nonlinearity of the operators being considered.

$$\begin{aligned} \text{Delay}((AB)_{kj}) &= \text{Delay}\big(g(\cup_{i=1}^n A_{ki}B_{ij})\big) \\ &\geq \min_i \big\{\text{Delay}(A_{ki}B_{ij})\big\} \end{aligned}$$

In other words, g would typically be a summation, but even if it is a more complicated function, the result still cannot be any faster than its fastest component.

We suppose that there are n subsystems, each with its own controller, and thus partition the sensor measurements and control actions as

$$y = \begin{bmatrix} y_1^T & \dots & y_n^T \end{bmatrix}^T \qquad u = \begin{bmatrix} u_1^T & \dots & u_n^T \end{bmatrix}^T$$

and then partition G and K as

$$G = \begin{bmatrix} G_{11} & \dots & G_{1n} \\ \vdots & & \vdots \\ G_{n1} & \dots & G_{nn} \end{bmatrix} \qquad K = \begin{bmatrix} K_{11} & \dots & K_{1n} \\ \vdots & & \vdots \\ K_{n1} & \dots & K_{nn} \end{bmatrix}$$

2.1 Propagation Delays

For any pair of subsystems i and j we define the propagation delay p_{ij} as the amount of time before a controller action at subsystem j can affect an output at subsystem i as such

$$p_{ij} \;=\; \mathrm{Delay}(G_{ij}) \qquad \forall\, i,j \in 1,\ldots,n$$

2.2 Transmission Delays

For any pair of subsystems k and l we define the (total) transmission delay t_{kl} as the minimum amount of time before the controller of subsystem k may use outputs from subsystem l. Given these constraints, we can define the overall subspace of admissible controllers S such that $K \in S$ if and only if

$$\mathrm{Delay}(K_{kl}) \;\geq\; t_{kl} \qquad \forall\, k,l \in 1,\ldots,n$$

In Section 3.1 we will break these total transmission delays out into a pure transmission delay, representing the time it takes to communicate the information from one subsystem to another, and a computational delay, representing the time it takes to process the information before it is used by the controller.

Triangle inequality. For the main result of this paper, we will assume that the triangle inequality holds amongst the transmission delays, that is,

$$t_{ki} + t_{ij} \;\geq\; t_{kj} \qquad \forall\, k,i,j \in 1,\ldots,n$$

This is typically a very reasonable assumption for the following reasons. t_{kj} is defined as the minimum amount of time before controller k can use outputs from subsystem j. Thus if there existed an i such that the inequality above failed, that would mean that controller k could receive that information more quickly if it came indirectly via controller i. We would thus reroute this information to go through i, t_{kj} would be reset to $t_{ki} + t_{ij}$, and the inequality would hold.

2.3 Problem Formulation

Given a plant G and transmission delays t_{kl} for each pair of subsystems, we define S as above, and we would then like to parameterize the following set

$$\{K : \mathcal{L}_e \rightarrow \mathcal{L}_e \mid K \text{ f.g. stabilizes G},\; K \in S\}$$

This gives all of the admissible controllers which are causal and stabilizing. The delays associated with dynamics propagating from one subsystem to another are embedded in G. The subspace of admissible controllers, S, has been defined to encapsulate the constraints on how quickly information may be passed from one subsystem to another. We call the subspace S the ***information constraint***.

Many constrained or decentralized control problems may be expressed in this form. In this paper, we focus on the case where S is defined by delay constraints as discussed above.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, a parameterization could be achieved with a simple change of variables [3, 1].

2.4 Invariance Condition

In this subsection we define an invariance condition pertaining to information constraints, and give a brief overview of results regarding this condition, in particular, that it allows parameterization of all causal stabilizing decentralized controllers.

We say (until a better name is established) that a set S is (1)-invariant under G if

$$K_1(I \pm GK_2) \in S \qquad \forall\, K_1, K_2 \in S \tag{1}$$

The following is the main result of [5]. It shows that constraints satisfying this condition are invariant under feedback.

Theorem 1. *Suppose that S is closed and that S is (1)-invariant under G. Then*

$$\big\{K(I - GK)^{-1} \mid K \in S\big\} \;=\; S$$

It was further shown in [5] that this result could be combined with those of [3, 1] to obtain the desired parameterization, when the plant is stable or unstable, respectively, as follows.

Suppose that G is inc. stable.

Then if S is closed and S is (1)-invariant under G, then we get the following parameterization.

$$\begin{aligned}&\{K : \mathcal{L}_e \rightarrow \mathcal{L}_e \mid K \text{ f.g. stabilizes G},\; K \in S\} \;=\\ &\big\{Q(I + GQ)^{-1} \mid Q : \mathcal{L}_e \rightarrow \mathcal{L}_e,\; Q \text{ f.g. stable},\; Q \in S\big\}\end{aligned}$$

We can generalize this to the case where the plant is unstable, provided that S is also a subspace, which it is for the delay constraints considered in this paper, and provided that an admissible controller exists which is both stable and stabilizing, i.e., if the plant is strongly stabilizable.

Suppose that $K_{\text{nom}} \in S$ is inc. stable, K_{nom} f.g. stabilizes G, and

$$\tilde{G} \;=\; G(I - K_{\text{nom}}G)^{-1}$$

is inc. stable.

For the remainder of this paper, we assume that for a given S, that K_{nom} and $\tilde{G}$ are defined in this manner.

Then if S is a closed subspace, and S is (1)-invariant under $\tilde{G}$, then

$$\begin{aligned}&\big\{K : \mathcal{L}_e \rightarrow \mathcal{L}_e \mid K \text{ f.g. stabilizes G},\; K \in S\big\} \;=\\ &\qquad\big\{K_{\text{nom}} + Q(I + \tilde{G}Q)^{-1} \mid\\ &\qquad\quad Q : \mathcal{L}_e \rightarrow \mathcal{L}_e,\; Q \text{ f.g. stable},\; Q \in S\big\}\end{aligned}$$

The main focus of this paper is thus characterizing delays for which S is (1)-invariant under G and $\tilde{G}$.

3 Conditions for Parameterization

We first provide a sufficient condition for Condition (1) to hold in terms of these delays.

Theorem 2. *Suppose that G and S are defined as above. S is (1)-invariant under G if*

$$t_{ki} + p_{ij} + t_{jl} \;\geq\; t_{kl} \qquad \forall\; i,j,k,l \tag{2}$$

Proof. Suppose $K_1, K_2 \in S$. First, for any i, l,

$$\begin{aligned} &\mathrm{Delay}\big((GK_2)_{il}\big) \\ &\quad \geq\; \min_j \{\mathrm{Delay}(G_{ij}) + \mathrm{Delay}((K_2)_{jl})\} \\ &\quad \geq\; \min_j \{p_{ij} + t_{jl}\} \end{aligned}$$

Noting that

$$\mathrm{Delay}(I_{il}) \;=\; \begin{cases} \infty & \text{if } \; i \neq l \\ 0 & \text{if } \; i = l \end{cases}$$

we then have

$$\begin{aligned} &\mathrm{Delay}\big((I \pm GK_2)_{il}\big) \\ &\quad \geq\; \min\{\mathrm{Delay}(I_{il}), \mathrm{Delay}((GK_2)_{il})\} \\ &\quad \geq\; \begin{cases} \min_j \{p_{ij} + t_{jl}\} & \text{if } \; i \neq l \\ 0 & \text{if } \; i = l \end{cases} \end{aligned}$$

and then for any k, l,

$$\begin{aligned} &\mathrm{Delay}\big((K_1(I \pm GK_2))_{kl}\big) \\ &\quad \geq\; \min_i \big\{\mathrm{Delay}((K_1)_{ki}) + \mathrm{Delay}\big((I \pm GK_2)_{il}\big)\big\} \\ &\quad \geq\; \min\Big\{\min_{i \neq l, j}\{t_{ki} + p_{ij} + t_{jl}\},\; t_{kl}\Big\} \\ &\quad =\; \min\Big\{\min_{i,j}\{t_{ki} + p_{ij} + t_{jl}\},\; t_{kl}\Big\} \end{aligned}$$

So if Condition (2) holds, then

$$\mathrm{Delay}\big((K_1(I \pm GK_2))_{kl}\big) \;\geq\; t_{kl} \quad \forall\; k,l$$

and thus

$$K_1(I \pm GK_2) \in S$$

■

Main Result. The following is the main result of this paper. It states that if the transmission delays satisfy the triangle inequality, and if the propagation delay between any pair of subsystems is at least as large as the transmission delay between those subsystems, then the information constraint is (1)-invariant. In other words, if along any link, data can be transmitted faster than dynamics propagate, then all causal stabilizing decentralized controllers can be parameterized.

Theorem 3. *Suppose that G and S are defined as above, and that the transmission delays satisfy the triangle inequality. If*

$$p_{ij} \;\geq\; t_{ij} \qquad \forall\; i,j \tag{3}$$

then S is (1)-invariant under G. Further, given $K_{\mathrm{nom}} \in S$, S is (1)-invariant under $\tilde{G}$.

Proof. Suppose Condition (3) holds. Then for all i, j, k, l we have

$$\begin{aligned} t_{ki} + p_{ij} + t_{jl} \;&\geq\; t_{ki} + t_{ij} + t_{jl} \\ &\geq\; t_{kl} \quad \text{by the triangle inequality} \end{aligned}$$

and thus by Theorem 2, S is (1)-invariant under G.

Now define

$$\check{S} \;=\; \{G : \mathcal{L}_e \to \mathcal{L}_e \mid \mathrm{Delay}(G_{ij}) \geq t_{ij} \quad \forall\; i,j\}$$

The new notation is introduced only because the dimensions may differ from those of S. It is clear from the above that S is (1)-invariant under any such G. Given $G_1, G_2 \in \check{S}$, and following similar steps to those in the proof of Theorem 2, for any k, l,

$$\begin{aligned} &\mathrm{Delay}\big((G_1(I \pm K_{\mathrm{nom}}G_2))_{kl}\big) \\ &\quad \geq\; \min\Big\{\min_{i,j}\{t_{ki} + t_{ij} + t_{jl}\},\; t_{kl}\Big\} \\ &\quad \geq\; t_{kl} \end{aligned}$$

and so $\check{S}$ is (1)-invariant under K_{nom}.

Then, by Theorem 1, $\tilde{G} \in \check{S}$, and thus S is (1)-invariant under $\tilde{G}$. ■

3.1 Computational Delays

In this section, we consider what happens when the controller of each subsystem has a computational delay c_i associated with it. The delay for controller i to use outputs from subsystem j, the total transmission delay, is then broken up into a pure transmission delay and this computational delay, as follows

$$t_{ij} = c_i + \tilde{t}_{ij}$$

If we were to assume that the triangle inequality held for the total transmission delays t_{ij} as before, then we would simply get the same results as in the previous section with the substitution above. In particular, we would find $p_{ij} \geq c_i + \tilde{t}_{ij}$ to be the condition for (1)-invariance. However, there are many cases where it makes sense to instead assume that the triangle inequality holds for the pure transmission delays $\tilde{t}_{ij}$, which is a stronger assumption.

In this section we derive conditions for (1)-invariance when we can assume that the triangle inequality holds for the pure transmission delays $\tilde{t}_{ij}$.

As before, the propagation delays are defined as

$$p_{ij} \;=\; \text{Delay}(G_{ij}) \qquad \text{for all } i,j$$

and S is now defined such that $K \in S$ if and only if

$$\text{Delay}(K_{kl}) \;\geq\; c_k + \tilde{t}_{kl} \qquad \text{for all } k,l$$

Thus the condition from Theorem 2 becomes

$$c_k + \tilde{t}_{ki} + p_{ij} + c_j + \tilde{t}_{jl} \;\geq\; c_k + \tilde{t}_{kl} \qquad \forall\; i,j,k,l$$

which reduces to

$$\tilde{t}_{ki} + p_{ij} + c_j + \tilde{t}_{jl} \;\geq\; \tilde{t}_{kl} \qquad \forall\; i,j,k,l \tag{4}$$

The following theorem gives conditions under which the information constraint is (1)-invariant. It states that if the triangle inequality holds amongst the pure transmission delays, and if Condition (5) holds, then the information constraint is (1)-invariant. Surprisingly, we see that the computational delay now appears on the left side of the inequality. In other words, not only does transmitting data faster than dynamics propagate still allow for parameterization when we account for computational delay, but the condition is actually relaxed.

Theorem 4. *Suppose that G and S are defined as above, and that the pure transmission delays satisfy the triangle inequality. If*

$$p_{ij} + c_j \;\geq\; \tilde{t}_{ij} \qquad \forall\; i,j \tag{5}$$

then S is (1)-invariant under G. Further, given $K_{\text{nom}} \in S$, S is (1)-invariant under $\tilde{G}$.

Proof. Suppose Condition (5) holds. Then for all i,j,k,l we have

$$\begin{aligned}\tilde{t}_{ki} + p_{ij} + c_j + \tilde{t}_{jl} \;&\geq\; \tilde{t}_{ki} + \tilde{t}_{ij} + \tilde{t}_{jl} \\ &\geq\; \tilde{t}_{kl} \quad \text{by the triangle inequality}\end{aligned}$$

and thus Condition (4) holds and S is (1)-invariant under G. This completes the proof for the stable case.

We can then similarly define

$$\check{S} \;=\; \{G : \mathcal{L}_e \to \mathcal{L}_e \mid \text{Delay}(G_{ij}) \geq \tilde{t}_{ij} - c_j \quad \forall\; i,j\}$$

so that S is clearly (1)-invariant under any such G. Given $G_1, G_2 \in \check{S}$, and following similar steps to those in the proof of Theorem 2, for any k,l,

$$\begin{aligned}&\text{Delay}\big((G_1(I \pm K_{\text{nom}}G_2))_{kl}\big) \\ &\quad\geq\; \min\Big\{\min_{i,j}\{\tilde{t}_{ki} + \tilde{t}_{ij} + \tilde{t}_{jl} - c_l\},\; \tilde{t}_{kl} - c_l\Big\} \\ &\quad\geq\; \tilde{t}_{kl} - c_l\end{aligned}$$

and so $\check{S}$ is (1)-invariant under K_{nom}.

Then, by Theorem 1, $\tilde{G} \in \check{S}$, and thus S is (1)-invariant under $\tilde{G}$. ∎

Thus we have shown that the triangle inequality and Condition (5) are sufficient for Condition (1), that is, for parameterization of causal stabilizing decentralized controllers.

4 Conclusions

We have studied the problem of parameterizing all stabilizing causal controllers for multiple subsystems subject to constraints on how quickly they can share information. In Theorem 3 we showed that, presuming the transmission delays satisfy the triangle inequality, if the transmission delay between any pair of subsystems is less than the corresponding propagation delay, then the information constraint is (1)-invariant. This allows for parameterization of all causal stabilizing controllers.

We further showed that if we separately account for computational delays, we still find that communicating faster than dynamics propagate along any link allows for this parameterization, and in fact, the condition becomes relaxed.

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