A Convex Characterization of Multidimensional Linear Systems Subject to SQI Constraints

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***Abstract*—This technical note proposes a convex characterization of the set of all stable closed-loop linear systems that are obtained from a given plant, which may be multidimensional, by interconnecting it in feedback with controllers that satisfy a certain pre-selected constraint. We take an approach that is particularly useful when a doubly-coprime factorization of the plant is difficult to obtain, and a stable stabilizing controller may not exist (the plant is not strongly-stabilizable) or when one may be difficult to find; most related work requires one of these. We adopt the so-called *coordinate-free* approach, which, unlike Youla's parametrization, does not rely on a doubly-coprime factorization of the plant. We show that if constraints which satisfy a condition called *strong quadratic invariance (SQI)* are imposed on the controllers then the set of all stable closed-loop multidimensional linear systems has a convex representation, and norm-optimal control problems can be cast in convex form. Although the SQI condition is in general slightly stronger than quadratic invariance (QI), which was developed in related work, they are equivalent for common classes of problems arising in decentralized control.**

***Index Terms*—Convex, decentralized control, linear Feedback Control Systems, multidimensional systems, optimal Control.**

I. Introduction

The design of norm-optimal structurally-constrained feedback systems is, in general, a hard problem, partly due to the lack of convexity that results from the interaction constraints [16] associated with the restrictions imposed on the controller. The theory developed in [10] characterizes cases for which the design of norm-optimal structurally-constrained controllers is tractable. In particular, [10] introduces a convex parametrization of all stabilizing controllers whose existence is determined by an algebraic test called quadratic invariance (QI), which takes into account the plant and the constraints imposed on the controller. The approach in [10] is valid for strongly–stabilizable plants; that is, plants for which a controller exists that is both stable and stabilizing. Another condition was developed for plants which are not necessarily linear or time-invariant [12], which also applied to strongly–stabilizable plants, using a parametrization from [1]. Recent work in [11] shows, for the unidimensional[1] LTI case, that it is possible to obtain a convex parametrization for all stabilizing controllers constrained to a QI subspace, even when the plant is not strongly–stabilizable. The method in [11] uses a structured doubly-coprime factorization of the plant with which Youla's classical parametrization is extended to account for QI subspace constraints on the controller. Recent results in [18], [19] have addressed the issue of structured state-space realizability of certain classes of structured-constrained controllers.

The main contribution of this technical note is to provide a new convex characterization of all closed-loop maps and associated controllers when these are restricted to a preselected *strongly quadratically invariant* (**SQI**) set. Following [14], we consider that the SQI set is a module over a ring of proper rational multidimensional transfer functions. As is shown in [14, Section VI], this framework is general enough to deal with sparsity and delay constraints, and it has the advantage that certain key invariance properties can be established without the need for intricate topology considerations. It is immediate from their definitions that if a module is SQI then it is also QI, and the converse also holds in many cases of interest, such as when it represents sparsity or delay constraints. In comparison to prior results, our approach has the following advantages: **i)** Instead of relying on a structured doubly-coprime factorization of the plant, our method only needs an initial admissible stabilizing controller. (see Section I-A) **ii)** The aforementioned initial controller does not need to be stable, which implies that our method is valid for non-strongly stabilizable plants. Our approach is based on the coordinate-free method [3]–[9], which was originally devised in an unconstrained setting, and here is generalized to allow SQI structural constraints on the controller.

A. Comparison with the Youla-Like Method in [11]

In comparison with [11], for the unidimensional case, the characterization proposed here requires parameters specified by larger matrices (see Remark III.3) and assumes that the constraints are encoded in an SQI module, as opposed to the more general QI subspaces allowed in [11]. However, the coordinate-free approach adopted here has the advantage that it does not rely on the intricate constrained doubly-coprime factorization needed in [11]. Although [15] has shown that every stabilizable multidimensional system has a doubly-coprime factorization, it also drew attention to the difficulty of computing one in practice. As a result, the characterization proposed here may be more suitable to deal not only with certain unidimensional systems, but also with multidimensional systems for which the type of constrained doubly-coprime factorization required in [11] has not been shown to exist and even if it does it would be no less difficult to obtain than the unconstrained case considered in [15]. Our approach can also account for delays and may apply to cases when the plant and controller are restricted to an integral domain, for which a coprime factorization does not exist [2].

II. Preliminaries

We adopt the algebraic formulation proposed in [14] that considers linear multidimensional systems represented by transfer functions that are real rational functions of d and s, which denote a vector of delay

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[1]Unidimensional refers to the standard case in which there is only one independent time variable. Unidimensional systems can have multiple inputs and outputs.

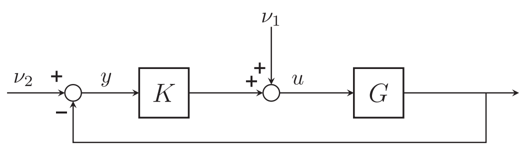

Fig. 1. Standard unity feedback interconnection.

operators $(d_1, \ldots, d_{n_d})$ and a vector of *transform domain* variables $(s_1, \ldots, s_{n_s})$, respectively. The authors of [14] use $\mathbb{R}(s, d)$ to denote the commutative ring with identity that is obtained from the aforementioned set by selecting the elements that are proper with respect to each entry of s. Multiplication and addition in $\mathbb{R}(s, d)$ are carried out using the standard rules for rational functions. Notice that using vector-valued d we can consider n_d different delays, and vector-valued s allows multidimensional transfer functions in the Laplace (or Z) domain that represent systems with n_s independent variables[2] that can be continuous (or discrete).

Henceforth, we assume that s and d are pre-selected and we use $\mathbb{P}$ to denote the subset of $\mathbb{R}(s, d)$ whose elements are proper in each entry of d. The subset of $\mathbb{P}$ whose elements are strictly proper in each entry of s is denoted with[3] $\mathbb{P}_s$. We assume that $\mathbb{P}$ and $\mathbb{P}_s$ inherit their sum and multiplication operators from $\mathbb{R}(s, d)$, which makes them a commutative ring with identity and a commutative ring, respectively.

Definition II.1: Consider that $\mathbb{A}$ is a pre-selected commutative subring of $\mathbb{P}$ whose elements we call stable. An example of a valid choice for $\mathbb{A}$ is the set of all elements of $\mathbb{P}$ that correspond to bounded-input bounded-output systems. A comprehensive discussion of stability concepts for multidimensional systems is given in [17].

By a natural extension of notation, we use $\mathbb{P}^{m\times n}$, $\mathbb{P}_s^{m\times n}$ and $\mathbb{A}^{m\times n}$ to denote the sets of matrices with m rows by n columns whose entries are in $\mathbb{P}$, $\mathbb{P}_s$ and $\mathbb{A}$, respectively. Matrix multiplication and addition within each set are carried out in the underlying ring that contains the entries. We refer to any matrix with entries in $\mathbb{P}$ as *TFM*, which stand for (multidimensional) transfer function matrix.

A. Feedback Stabilization of a Generalized Plant: Notation and Framework

We consider the feedback system displayed in Fig. 1, in which G and K represent a plant and a feedback controller, respectively.

Assumption II.2 (Well-Posedness): From this point onward, we assume that K is in $\mathbb{P}^{n_u \times n_y}$ and G is in $\mathbb{P}_s^{n_y \times n_u}$.

From the proof of [14, Theorem 10], we conclude that Assumption II.2 suffices to guarantee that $(I - KG)$ and $(I - GK)$ have well-defined inverses in $\mathbb{P}^{n_u \times n_u}$ and $\mathbb{P}^{n_y \times n_y}$, respectively. We then proceed to define $\mathcal{H}(G, K)$ as the TFM from $[\nu_2^T \quad \nu_1^T]^T$ to $[y^T \quad u^T]^T$, which can be written as

$$\mathcal{H}(G, K) \overset{def}{=} \begin{bmatrix} (I - GK)^{-1} & G(I - KG)^{-1} \\ K(I - GK)^{-1} & (I - KG)^{-1} \end{bmatrix}. \qquad (1)$$

If the entries of $\mathcal{H}(G, K)$ are in $\mathbb{A}$, we say that K is a *stabilizing controller* for G or equivalently that K *stabilizes* G.

[2]The unidimensional case, in which time is the sole independent variable, is characterized by $n_s = 1$.

[3]The sets $\mathbb{P}$ and $\mathbb{P}_s$ used here are subsets of $\mathbb{R}(s, d)_p$ and $\mathbb{R}(s, d)_{sp}$ as defined in [14], respectively.

III. The Coordinate-Free Approach

This section gives a brief overview of the results in [6]–[9], which develop the so-called *coordinate-free approach* to linear control design. Our discussion emphasizes concepts that are used throughout this technical note.

A. Convex Characterization of All Stable Closed-Loop TFMs

For a given G, the coordinate-free approach ([6]–[9]) yields a convex parametrization of the set of closed-loop TFMs $\mathcal{H}(G, K)$ that can be achieved by a stabilizing controller K. Unlike Youla's classical method, it does so without a doubly-coprime factorization of the plant, but it requires prior knowledge of a stabilizing controller. The coordinate-free method was pursued in [3]–[9], as a viable approach for when a stabilizing controller is known and the coprime factorizability of the plant is not, either because it is nonexistent or difficult to compute (e.g., for multidimensional systems).

Assumption III.1: From this point onwards, we assume that G in $\mathbb{P}_s^{n_y \times n_u}$ is given and fixed. Hence, we will simplify our notation by omitting G from the definition of sets and maps even when they depend on G.

We define the set of all stable closed-loop TFMs as follows:

$$\mathbb{H} \overset{def}{=} \left\{ \mathcal{H}(G, K) \in \mathbb{A}^{(n_u + n_y)\times(n_u + n_y)} \,\middle|\, K \in \mathbb{P}^{n_u \times n_y} \right\}.$$

The following theorem is derived from statements in [7, Propo. 4 and 5], and will be instrumental in the sequel. It establishes a convex parametrization of all stable closed-loop maps, and associated stabilizing controllers.

Theorem III.2: Suppose that K_0 in $\mathbb{P}^{n_u \times n_y}$ is a controller that stabilizes G.

1) [7, Proposition 4] The following equality holds:

$$\left\{ \Omega(Q, K_0) \,\middle|\, Q \in \mathbb{A}^{(n_u + n_y)\times(n_u + n_y)} \right\} = \mathbb{H}$$

where, for Q in $\mathbb{A}^{(n_u + n_y)\times(n_u + n_y)}$, $\Omega(Q, K_0)$ is defined as

$$\Omega(Q, K_0) \overset{def}{=} \mathcal{H}(G, K_0) + \left(\mathcal{H}(G, K_0) - \begin{bmatrix} I_{n_y} & 0 \\ 0 & 0 \end{bmatrix} \right) \times Q \left(\mathcal{H}(G, K_0) - \begin{bmatrix} 0 & 0 \\ 0 & I_{n_u} \end{bmatrix} \right). \qquad (2)$$

Here, I_{n_y} and I_{n_u} denote the identity matrices of dimension n_y and n_u, respectively.

2) [7, Proposition 5] The following equality holds:

$$\left\{ \mathcal{H}(G, \mathcal{K}(Q, K_0)) \,\middle|\, Q \in \mathbb{A}^{(n_y + n_u)\times(n_y + n_u)} \right\} = \mathbb{H} \qquad (3)$$

where $\mathcal{K}(Q, K_0)$ is the controller defined as

$$\mathcal{K}(Q, K_0) \overset{def}{=} \Omega_{21}(Q, K_0)\Omega_{11}^{-1}(Q, K_0) \qquad (4)$$

$$= \Omega_{22}^{-1}(Q, K_0)\Omega_{21}(Q, K_0) \qquad (5)$$

and where we adopt the following partition for $\Omega(Q, K_0)$:

$$\Omega(Q, K_0) = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad}^{n_y} & \overbrace{\qquad\qquad}^{n_u} \end{array} \\ \begin{bmatrix} \Omega_{11}(Q, K_0) & \Omega_{12}(Q, K_0) \\ \Omega_{21}(Q, K_0) & \Omega_{22}(Q, K_0) \end{bmatrix} \end{array} \begin{array}{l} \}n_y \\ \}n_u. \end{array} \qquad (6)$$

Furthermore, for every $Q \in \mathbb{A}^{(n_u + n_y)\times(n_u + n_y)}$, the controller $\mathcal{K}(Q, K_0)$ is proper.

Remark III.3: Note that the dimensions of the parameter Q are not those of the associated controller $\mathcal{K}(Q, K_0)$, as they would be in Youla's classical parametrization; instead, our version of Q has the same number of columns and rows as $\mathcal{H}(G,K)$.

B. The Disturbance Attenuation Problem via the Coordinate-Free Approach

The following is a standard norm-optimal control problem:

$$\underset{\substack{K \text{ stabilizes } G\\ K\,\in\,\mathbb{P}^{n_u\times n_y}}}{\text{minimize}} \quad \|\mathcal{F}(P,K)\| \tag{7}$$

where $\mathcal{F}(P,K)$, which represents the lower-linear fractional transformation of a given proper generalized plant P with controller K, is defined as follows:

$$\mathcal{F}(P,K) \stackrel{def}{=} P_{zw} + P_{zu}\, K\big(I-GK\big)^{-1} P_{yw}. \tag{8}$$

Here, $\mathcal{F}(P,K)$ can be viewed as the TFM from an input w to a performance output z, while $P_{zw}\in\mathbb{P}^{n_z\times n_w}$, $P_{zu}\in\mathbb{P}^{n_z\times n_u}$, $P_{yw}\in\mathbb{P}^{n_y\times n_w}$ and $G\in\mathbb{P}_s^{n_y\times n_u}$ represent the components of P relating w and z with u and y. In (7), $\|\cdot\|$ represents a pre-selected norm over a vector space that embeds $\mathbb{P}^{n_z\times n_w}$.

The following result shows that $\mathcal{F}(P,K)$ can be expressed as an affine map of the parameter Q. The proof is outlined in [9, Section IV].

Theorem III.4: If K_0 in $\mathbb{P}^{n_u\times n_y}$ is a controller that stabilizes G then the following holds for all Q in $\mathbb{A}^{(n_u+n_y)\times(n_u+n_y)}$:

$$\mathcal{F}\big(P,\mathcal{K}(Q,K_0)\big) \;=\; T_1 - T_2 Q T_3, \tag{9}$$

where T_1, T_2, and T_3, are the TFMs defined as

$$\begin{aligned} T_1 &\stackrel{def}{=} P_{zw} + P_{zu}\, K_0(I-GK_0)^{-1}P_{yw},\\ T_2 &\stackrel{def}{=} \big[\, P_{zu}K_0(I-GK_0)^{-1} \quad P_{zu}(I-K_0G)^{-1}\,\big],\\ T_3 &\stackrel{def}{=} \begin{bmatrix} (I-GK_0)^{-1}P_{yw}\\ K_0(I-GK_0)^{-1}P_{yw}\end{bmatrix}. \end{aligned} \tag{10}$$

Remark III.5: The identity in (3) and Theorem III.4 imply that the following holds:

$$\begin{aligned}&\{\mathcal{F}(P,K)\,|\,K\in\mathbb{P}^{n_u\times n_y},\ K \text{ stabilizes } G\}\\ &\qquad = \{T_1 - T_2QT_3\,|\,Q\in\mathbb{A}^{(n_u+n_y)\times(n_u+n_y)}\}\end{aligned} \tag{11}$$

and the standard problem in (7) is equivalent to the following model-matching problem [9]:

$$\underset{Q\in\mathbb{A}^{(n_u+n_y)\times(n_u+n_y)}}{\text{minimize}} \quad \|\, T_1 \;-\; T_2QT_3\|. \tag{12}$$

More specifically, problems (7) and (12) achieve the same value, and Q^* is a solution of (12) if and only if $K^*=\mathcal{K}(Q^*,K_0)$ is a solution of (7).

Remark III.6: Since we use [9] throughout this technical note, we need to note that there is a typo in [9, Section III]. Namely, the expression for the controller in [9] is given as $\Omega_{21}(Q,K_0)\Omega_{22}^{-1}(Q,K_0)$, but it should have been $\Omega_{22}^{-1}(Q,K_0)\Omega_{21}(Q,K_0)$, as we use in our definition of $\mathcal{K}(Q,K_0)$.

IV. Control Design Subject to SQI Constraints

Henceforth, we consider that there are structural constraints on the controller, and we use $\mathbb{S}$ to denote the set of admissible controllers. We follow the framework of [14] in which $\mathbb{S}$ is both a subset of $\mathbb{P}^{n_u\times n_y}$ and a $\mathbb{P}$-module. As is evident from [14], the fact that $\mathbb{S}$ is a $\mathbb{P}$-module enables the study of key invariance properties without the need for intricate topology considerations. The merits and limitations of this approach, in comparison to [11] and [10], are further discussed in Section I-A and [14], respectively. Examples in [14, Section VI] show how to specify $\mathbb{S}$ to impose sparsity and delay constraints on the controller.

A. Strong Quadratic Invariance (SQI)

Throughout this technical note, we will consider that $\mathbb{S}$ is given and fixed, and we will also assume that it is strongly quadratically invariant (SQI) with respect to G.

The definition and basic properties of SQI are given below.

Definition IV.1: $\mathbb{S}$ is strongly quadratically invariant (SQI) under G if and only if the following invariance condition holds:

$$KGJ\in\mathbb{S},\quad \text{for all}\quad K,J\ \in\mathbb{S}.$$

Remark IV.2: The SQI concept is inspired on the more general notion of quadratic invariance (QI) proposed in [10]. It follows immediately from their definitions that, for a given G, if $\mathbb{S}$ is SQI then it is also QI, and the converse also holds in many cases of interest, such as for sparsity and delay constraints. The QI/SQI equivalence[4] for sparsity constraints is shown in [10, Theorem 26], while the QI/SQI equivalence for delay constraints is clear from the proofs in [13].

The symmetric control problem, where the plant is symmetric, and the controller to be designed must be symmetric as well, is an example of a problem that is QI but not SQI.

The following proposition is a slight adaptation of [10, Lemma 5], which, as discussed after the proof of [14, Lemma 11], can be proved using a simple induction argument.

Proposition IV.3: If $\mathbb{S}$ is SQI with respect to G, then the following hold for any non-negative integer n:

$$J(GK)^n\in\mathbb{S},\quad \text{for all}\quad K,J\ \in\mathbb{S}, \tag{13}$$

$$(KG)^nJ\in\mathbb{S},\quad \text{for all}\quad K,J\ \in\mathbb{S}. \tag{14}$$

The following lemma is important throughout technical note.

Lemma IV.4: The following inclusions hold for any $\mathbb{S}$ that is SQI with respect to G:

$$J(I-GK)^{-1}\in\mathbb{S}\quad \text{for all } K,J\in\mathbb{S}, \tag{15}$$

$$(I-KG)^{-1}J\in\mathbb{S}\quad \text{for all } K,J\in\mathbb{S}. \tag{16}$$

Proof: We follow the same line of proof adopted for [14, Theorems 9 and 10]. As it is argued in [14, Theorem 10], Assumption II.2 guarantees that $(I-GK)^{-1}$ is a well-defined element of $\mathbb{P}^{n_y\times n_y}$. We can then follow the approach of [14, Theorem 6] to conclude that, for our G in $\mathbb{P}_s^{n_y\times n_u}$ and for any K in $\mathbb{P}^{n_u\times n_y}$ there exist weights $\{\alpha_{i,(G,K)}\}_{i=1}^{n_y}$ in $\mathbb{P}$, such that for any $K,J\in\mathbb{P}^{n_u\times n_y}$, the following holds:

$$J(I-GK)^{-1} \;=\; \sum_{i=1}^{n_y}\alpha_{i,(G,K)}J(GK)^{i-1}. \tag{17}$$

Now assuming that $K,J\in\mathbb{S}$, we apply Proposition IV.3 to (17) and the assumption that $\mathbb{S}$ is a $\mathbb{P}$-module to conclude that (15) holds. The proof of (16) follows an analogous procedure along with (14). ■

B. Norm-Optimal Control Design Subject to SQI

Henceforth, we are concerned with the following problem.

[4]It should also be noted that the equivalences between the SQI and QI conditions for sparsity and delay constraints, as stated in here, are based on algebraic manipulations that hold regardless of whether $\mathbb{S}$ is a $\mathbb{P}$-module or a subspace.

Problem IV.5: Let $\mathbb{S}$ be SQI with respect to G. Consider the following constrained version of (7):

$$\underset{K \in \mathbb{K}}{\text{minimize}} \ \|\mathcal{F}(P, K)\| \tag{18}$$

where $\mathbb{K}$ is the set of admissible stabilizing controllers defined as[5]

$$\mathbb{K} \overset{def}{=} \{K \in \mathbb{S} \mid K \text{ stabilizes } G\}. \tag{19}$$

Remark IV.6: Notice that, depending on the choice of the norm for (18), a minimum may not be attained in $\mathbb{K}$. In fact, an example in [20] illustrates, for the H_∞ norm, that an optimal solution may be unique and irrational. From [10], we know that a minimum in $\mathbb{K}$ always exists for the H_2 norm.

C. A Convex Approach to Problem IV.5

We start by defining the noise sensitivity map $\mathcal{W} : \mathbb{P}^{n_u \times n_y} \rightarrow \mathbb{P}^{n_u \times n_y}$ as follows:

$$\mathcal{W}(V) \overset{def}{=} -V(I - GV)^{-1}, \tag{20}$$

It is straightforward to verify that (20) is an involution; that is, $\mathcal{W}(\mathcal{W}(V)) = V$ for any $V \in \mathbb{P}^{n_u \times n_y}$. It is also clear from (15), or [14, Theorem 10], that if $\mathbb{S}$ is SQI with respect to G, then $\mathbb{S}$ is invariant under $\mathcal{W}$.

The following lemma gives a convenient expression for $\mathcal{K}(Q, K_0)$, and also characterizes a domain set for Q for which the associated controllers are all admissible and stabilizing.

Lemma IV.7: Suppose that $\mathbb{S}$ is SQI with respect to G, and that K_0 is in $\mathbb{K}$.

1) The following holds for any Q in $\mathbb{A}^{(n_y+n_u)\times(n_y+n_u)}$:

$$\mathcal{K}(Q, K_0) = \mathcal{W}\big(-\Omega_{21}(Q, K_0)\big) \tag{21}$$

and $\Omega_{21}(Q, K_0)$ can be written as

$$\begin{aligned}\Omega_{21}(Q, K_0) = {}& \big(I - K_0 G\big)^{-1}\big(K_0 Q_{11} + K_0 Q_{12} K_0 + Q_{21} \\ & + Q_{22} K_0 + K_0 - K_0 G K_0\big)\big(I - G K_0\big)^{-1}\end{aligned} \tag{22}$$

where we adopt the following conformable partition of Q:

$$Q = \begin{array}{cc} \overbrace{\phantom{Q_{11}}}^{n_y} & \overbrace{\phantom{Q_{12}}}^{n_u} \\ \end{array}$$
$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{array}{l} \}n_y \\ \}n_u. \end{array} \tag{23}$$

2) The following inclusion holds:

$$Q \in \mathbb{T}_{K_0} \Rightarrow \mathcal{K}(Q, K_0) \in \mathbb{K} \tag{24}$$

where $\mathbb{T}_{K_0}$ is a convex set defined as

$$\begin{aligned}\mathbb{T}_{K_0} \overset{def}{=} {}& \{Q \in \mathbb{A}^{(n_y+n_u)\times(n_y+n_u)} \mid \\ & \big(K_0 Q_{11} + K_0 Q_{12} K_0 + Q_{21} + Q_{22} K_0\big) \in \mathbb{S}\}.\end{aligned} \tag{25}$$

Proof: The equality in (21) follows from (4) and the expression for $\Omega_{11}(Q, K_0)$ that is given in the Appendix in Proposition A.1. The expression in (22) also follows from Proposition A.1.

Now, suppose $Q \in \mathbb{T}_{K_0}$, and let $\bar{J} = K_0 Q_{11} + K_0 Q_{12} K_0 + Q_{21} + Q_{22} K_0 + K_0 - K_0 G K_0$, such that $\Omega_{21}(Q, K_0) = (I - K_0 G)^{-1} \bar{J} (I - G K_0)^{-1}$. We are given that $K_0 \in \mathbb{K} \subset \mathbb{S}$, it follows from SQI that $K_0 G K_0 \in \mathbb{S}$, and it follows from the definition of $\mathbb{T}_{K_0}$ that $K_0 Q_{11} + K_0 Q_{12} K_0 + Q_{21} + Q_{22} K_0 \in \mathbb{S}$, and thus $\bar{J} \in \mathbb{S}$. It then follows from (15) and (16) that $\Omega_{21}(Q, K_0) \in \mathbb{S}$. Lastly, since $\mathbb{S}$ is invariant under $\mathcal{W}$, it follows from (21) that $\mathcal{K}(Q, K_0) \in \mathbb{S}$, and thus (24) is proven. ■

We now proceed by defining a map which will take controllers back into the space of Q-parameters.

Definition IV.8: We define $\mathcal{Q} : \mathbb{P}^{n_u \times n_y} \rightarrow \mathbb{P}^{(n_y+n_u)\times(n_y+n_u)}$ as follows:

$$\mathcal{Q}(K) \overset{def}{=} \begin{bmatrix} G\mathcal{W}(K) - I & G - G\mathcal{W}(K)G \\ -\mathcal{W}(K) & \mathcal{W}(K)G \end{bmatrix}. \tag{26}$$

We now give a lemma showing that this map takes admissible stabilizing controllers to a particular convex set of Q-parameters, which can then be returned to the original controller with our previously defined map from Q-paramters to controllers.

Lemma IV.9: Suppose that $\mathbb{S}$ is SQI with respect to G and that K_0 is in $\mathbb{K}$. The following hold:

$$K \in \mathbb{K} \Rightarrow \mathcal{Q}(K) \in \mathbb{T}, \tag{27a}$$

$$K \in \mathbb{K} \Rightarrow \mathcal{K}(\mathcal{Q}(K), K_0) = K \tag{27b}$$

where the convex set $\mathbb{T}$ is defined as

$$\begin{aligned}\mathbb{T} \overset{def}{=} {}& \{Q \in \mathbb{A}^{(n_y+n_u)\times(n_y+n_u)} \mid \\ & (JQ_{11} + Q_{21} + JQ_{12}L + Q_{22}L) \in \mathbb{S}, \text{ for all } J, L \in \mathbb{S}\}.\end{aligned} \tag{28}$$

Proof: In order to prove (27a), we start by establishing the following partition of $\mathcal{Q}(K)$, in which we suppress the dependence on K for simplicity of notation

$$\mathcal{Q}_{11} \overset{def}{=} G\mathcal{W}(K) - I \quad = (I - GK)^{-1} \tag{29a}$$

$$\mathcal{Q}_{12} \overset{def}{=} G - G\mathcal{W}(K)G = G(I - KG)^{-1} \tag{29b}$$

$$\mathcal{Q}_{21} \overset{def}{=} -\mathcal{W}(K) \quad = K(I - GK)^{-1} \tag{29c}$$

$$\mathcal{Q}_{22} \overset{def}{=} \mathcal{W}(K)G \quad = I - (I - KG)^{-1} \tag{29d}$$

and in which we also establish equivalent expressions with repeated use of the push-through formula which will be useful.

From here on, assume that K is chosen in $\mathbb{K}$. We thus have $K \in \mathbb{S}$, and from (15) we conclude that $\mathcal{W}(K)$ and thus $\mathcal{Q}_{21}$ are both in $\mathbb{S}$. It then follows from SQI that for any $J, L \in \mathbb{S}$, the terms $J\mathcal{Q}_{11}$, $J\mathcal{Q}_{12}L$ and $\mathcal{Q}_{22}L$ are in $\mathbb{S}$. Since $\mathbb{S}$ is a $\mathbb{P}$-module, we then have $J\mathcal{Q}_{11} + \mathcal{Q}_{21} + J\mathcal{Q}_{12}L + \mathcal{Q}_{22}L \in \mathbb{S}$.

To prove (27a), it remains to show that $\mathcal{Q}(K) \in \mathbb{A}^{(n_y+n_u)\times(n_y+n_u)}$. Since $K \in \mathbb{K}$, we know that K stabilizes G, and thus that $\mathcal{H}(G, K) \in \mathbb{A}^{(n_y+n_u)\times(n_y+n_u)}$. Comparing this with the equivalent expressions for the blocks of $\mathcal{Q}(K)$ given in (29), we indeed have $\mathcal{Q}(K) \in \mathbb{A}^{(n_y+n_u)\times(n_y+n_u)}$.

Still assuming that $K \in \mathbb{K}$, we now establish (27b)

$$\begin{aligned}\mathcal{K}(\mathcal{Q}(K), K_0) &\overset{(21)}{=} \mathcal{W}\big(-\Omega_{21}(\mathcal{Q}(K), K_0)\big) \\ &\overset{(32)}{=} \mathcal{W}\big(\mathcal{W}(K)\big) \overset{(a)}{=} K\end{aligned}$$

where (a) follows from the fact that $\mathcal{W}$ is involutory. ■

Remark IV.10: We have $\mathbb{T} \subset \mathbb{T}_{K_0}$, since one could choose $J = L = K_0$ in (28). In lieu of (28), we could have equivalently defined

$$\begin{aligned}\mathbb{T} = {}& \{Q \in \mathbb{A}^{(n_y+n_u)\times(n_y+n_u)} \mid \\ & JQ_{11}, Q_{21}, JQ_{12}L, Q_{22}L \in \mathbb{S}, \text{ for all } J, L \in \mathbb{S}\},\end{aligned}$$

[5]It follows from [9, Lemma 1] that K stabilizes G if and only if it stabilizes the generalized plant P, with respect to the stability definition given in [9]. Hence, the set $\mathbb{K}$ could have been equivalently defined as $\{K \in \mathbb{S} \mid K \text{ stabilizes } P\}$. The stabilization of generalized plants is not discussed here due to space limitations.

as these are easily shown to be equivalent, and this may sometimes be more desirable to work with. If we were to similarly alter the definition of our other set of parameters to obtain

$$\widetilde{\mathbb{T}}_{K_0} = \{Q \in \mathbb{A}^{(n_y+n_u)\times(n_y+n_u)} \mid K_0 Q_{11}, Q_{21}, K_0 Q_{12} K_0, Q_{22} K_0 \in \mathbb{S}\},$$

we would in general have $\mathbb{T} \subset \widetilde{\mathbb{T}}_{K_0} \subset \mathbb{T}_{K_0}$.

Remark IV.11: The mapping to the controller with restricted domain and co-domain $\mathcal{K}(\cdot, K_0) : \mathbb{T} \to \mathbb{K}$ is surjective, since for any $K \in \mathbb{K}$ we can choose $Q = \mathcal{Q}(K)$, and then have $Q \in \mathbb{T}$ and $\mathcal{K}(Q, K_0) = K$ from (27a) and (27b), respectively.

This is a key fact in the proof of the following theorem. It is also interesting to note that (27b) holds regardless of the choice of K_0 in $\mathbb{K}$.

The following theorem and corollary are our main results. They establish a convex model-matching formulation for Problem IV.5.

Theorem IV.12: Suppose that $\mathbb{S}$ is SQI with respect to G, and that K_0 is in $\mathbb{K}$. The following holds:

$$\{\mathcal{F}(P, K) \mid K \in \mathbb{K}\} = \{T_1 - T_2 Q T_3 \mid Q \in \mathbb{T}\} \tag{30}$$

where T_1, T_2, T_3 are as given in (10).

Proof: We first establish "$\subset$" in (30). Given $K \in \mathbb{K}$, we choose $Q = \mathcal{Q}(K)$. From (27a), we then have $Q \in \mathbb{T}$, and then have the equalities $T_1 - T_2 Q T_3 \overset{(9)}{=} \mathcal{F}(P, \mathcal{K}(Q, K_0)) \overset{(27b)}{=} \mathcal{F}(P, K)$.

We now establish "$\supset$" in (30). Given $Q \in \mathbb{T}$, choose $K = \mathcal{K}(Q, K_0)$. Since $\mathbb{T} \subset \mathbb{T}_{K_0}$, it then follows from (24) that $K \in \mathbb{K}$. We then again have $\mathcal{F}(P, \mathcal{K}(Q, K_0)) \overset{(9)}{=} T_1 - T_2 Q T_3$, and (30) is proven. ■

The following corollary follows from Remark IV.11 and Theorem IV.12.

Corollary IV.13: Let $\mathbb{S}$ be SQI with respect to G, and K_0 be in $\mathbb{K}$. Consider the following problem:

$$\underset{Q \in \mathbb{T}}{\text{minimize}} \ \|T_1 - T_2 Q T_3\|. \tag{31}$$

Problem IV.5 and (31) have the same optimal value, and Problem IV.5 has an optimal solution if and only if (31) has an optimal solution. In addition, Q^* is optimal for (31) if and only if $K^* = \mathcal{K}(Q^*, K_0)$ is optimal for Problem IV.5.

D. Comparison Between Sets $\mathbb{T}_{K_0}$, $\widetilde{\mathbb{T}}_{K_0}$, and $\mathbb{T}$

It follows immediately from Remark IV.10 and from the proof of Theorem IV.12 that the set $\mathbb{T}$ could be replaced with $\mathbb{T}_{K_0}$ or $\widetilde{\mathbb{T}}_{K_0}$ in (30), and thus in Corollary IV.13 as well. As we point out in Remark IV.10, the set $\mathbb{T}$ is contained in $\widetilde{\mathbb{T}}_{K_0}$, which is contained in $\mathbb{T}_{K_0}$. This establishes a tradeoff between *size* versus *complexity* of representation because, while $\mathbb{T} \subset \widetilde{\mathbb{T}}_{K_0} \subset \mathbb{T}_{K_0}$ holds, the representation of $\mathbb{T}_{K_0}$ is simpler than that of $\widetilde{\mathbb{T}}_{K_0}$, which, in turn, is less intricate than that of $\mathbb{T}$. For a given plant and information structure, one thus has the latitude to select any of these sets to optimize over, after assessing the aforesaid tradeoff.

V. Conclusions

This technical note presents a new convex characterization of all closed-loop systems which result from applying stabilizing controllers that satisfy pre-selected constraints to a given plant. The coordinate-free approach is utilized, the advantages of which include not requiring a doubly-coprime factorization, which can be difficult to obtain for some multidimensional plants, nor requiring a stable stabilizing controller, which only exists for strongly stabilizable plants and may be difficult to systematically find even when it does exist. Much related work in optimal stabilizing constrained control requires one of these.

We introduce a condition relating the controller constraints to the plant called SQI, which is generally slightly stronger than QI, and equivalent for problem classes of interest in decentralized control. We show that when the constraints define a module and are SQI, the set of closed-loop systems that result from using admissible stabilizing controllers can be represented with affine constraints on the variable from the coordinate-free parametrization. This representation can be used to cast constrained norm-optimal control problems in convex form.

Appendix

The following proposition establishes a key algebraic identity to prove Lemma IV.7.

Proposition A.1: Let G in $\mathbb{P}_s^{n_y \times n_u}$ be given. The following identities hold for every Q in $\mathbb{A}^{(n_y+n_u)\times(n_y+n_u)}$ and for every K_0 in $\mathbb{P}^{n_u \times n_y}$:

$$\begin{aligned} \Omega_{11}(Q, K_0) &= I_{n_y} + G\Omega_{21}(Q, K_0), \\ \Omega_{21}(Q, K_0) &= (I - K_0 G)^{-1}(K_0 Q_{11} + K_0 Q_{12} K_0 + Q_{21} \\ &\quad + Q_{22} K_0 + K_0 - K_0 G K_0)(I - G K_0)^{-1}. \end{aligned}$$

Proof: The proof follows from (2) and the matrix inversion lemma (Woodbury formula) applied to the first block-column of $\Omega(Q, K_0)$. ■

Lemma A.2: Let G in $\mathbb{P}_s^{n_y \times n_u}$ be given. The following holds for all K and K_0 in $\mathbb{P}^{n_u \times n_y}$:

$$\Omega_{21}\big(\mathcal{Q}(K), K_0\big) = -\mathcal{W}(K). \tag{32}$$

Proof: For an arbitrary K in $\mathbb{K}$, substitution of (26) in (22) leads to

$$\begin{aligned} \Omega_{21}\big(\mathcal{Q}(K), K_0\big) &= (I - K_0 G)^{-1}(K_0 \mathcal{Q}_{11} + K_0 \mathcal{Q}_{12} K_0 + \mathcal{Q}_{21} \\ &\quad + \mathcal{Q}_{22} K_0 + K_0 - K_0 G K_0)(I - G K_0)^{-1} \end{aligned} \tag{33}$$

where we use $\mathcal{Q}_{ij}$ to indicate the blocks of $\mathcal{Q}(K)$ as defined in (29). Substitution of (29) in (33), and a few algebraic simplifications lead to (32). ■

References

[1] V. Anantharam and C. Desoer, "On the stabilization of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 29, pp. 569–572, 1984.

[2] V. Anantharam, "On stabilization and the existence of coprime factorization," *IEEE Trans. Autom. Control*, vol. 30, pp. 1030–1031, 1985.

[3] S. Shankar and V. R. Sule, "Algebraic geometric aspects of feedback stabilization," *SIAM J. Control Optim.*, vol. 30, no. 1, pp. 11–30, 1992.

[4] V. R. Sule, "Feedback stabilization over commutative rings: The matrix case," *SIAM J. Control Optim.*, vol. 32, pp. 1675–1695, 1994.

[5] V. R. Sule, "Corrigendum: Feedback stabilization over commutative rings: The matrix case," *SIAM J. Control Optim.*, vol. 36, pp. 2194–2195, 1998.

[6] K. Mori and K. Abe, "Feedback stabilization over commutative rings: Further study of coordinate-free approach," *SIAM J. Control Optim.*, vol. 39, no. 6, pp. 1952–1973, 2001.

[7] K. Mori, "Parameterization of stabilizing controllers over commutative rings with applications to multidimensional systems," *IEEE Trans. Circuits Syst. I*, vol. 49, pp. 743–752, 2002.

[8] K. Mori, "Elementary proof of controller parametrization without coprime factorizability," *IEEE Trans. Autom. Control*, vol. 49, no. 4, pp. 589–592, 2004.

[9] K. Mori, "Relationship between standard control problem and model-matching problem without coprime factorizability," *IEEE Trans. Autom. Control*, vol. 49, no. 2, pp. 230–233, 2004.

[10] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *IEEE Trans. Autom. Control*, vol. 51, no. 2, pp. 274–286, 2006.

[11] Ş. Sabău and N. C. Martins, "Youla-like parametrizations subject to QI subspace constraints," *IEEE Trans. Autom. Control*, vol. 59, no. 6, pp. 1411–1422, 2014.

[12] M. Rotkowitz, "Information structures preserved under nonlinear time-varying feedback," *Proc. American Control Conf.*, 2006, pp. 4207–4212.

[13] M. Rotkowitz, R. Cogill, and S. Lall, "Convexity of optimal control over networks with delays and arbitrary topology," *Int. J. Syst., Control, Commun.*, vol. 2, no. 1/2/3, 2010, pp. 30–54.

[14] L. Lessard and S. Lall, "An algebraic approach to the control of decentralized systems," *IEEE Trans. Control Network Syst.*, vol. 1, no. 4, 2014, pp. 308–317.

[15] A. Quadrat, "Every internally stabilizable multidimensional system admits a doubly coprime factorization," *Proc. Mathemat. Theory Networks Syst.*, 2004.

[16] R. A. Date and J. H. Chow, "Decentralized stable factors and a parametrization of decentralized controllers," *IEEE Trans. Autom. Control*, vol. 39, no. 2, pp. 347–351, 1994.

[17] N. K. Boise, Multidimensional Systems Theory and Applications. New York: Springer, 2nd ed., 2003.

[18] A. S. M. Vamsi and N. Elia, "Optimal distributed controllers realizable over arbitrary networks," *IEEE Trans. Autom. Control*, vol. 61, no. 1, pp. 129–144, 2016.

[19] L. Lessard, M. Kristalny and A. Rantzer, "On structured realizability and stability of linear systems," *Proc. American Control Conf.*, 2013, pp. 5784–5790.

[20] A. Megretski, "H-infinity optimal decentralized matching model is not always rational," arXiv:1305.5856v1, 2013.