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Fixed Modes of Decentralized Systems with Arbitrary Information Structure

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***Abstract*— A seminal result in decentralized control is the development of fixed modes by Wang and Davison in 1973 - that plant modes which cannot be moved with a static decentralized controller cannot be moved by a dynamic one either, and that the other modes which can be moved can be shifted to any chosen location with arbitrary precision. These results were developed for perfectly decentralized, or block diagonal, information structure, where each control input may only depend on a single corresponding measurement. Furthermore, the results were claimed after a preliminary step was demonstrated, omitting a rigorous induction for each of these results, and the remaining task is nontrivial.**

In this paper, we consider fixed modes for arbitrary information structures, where certain control inputs may depend on some measurements but not others. We provide a comprehensive proof that the modes which cannot be altered by a static controller with the given structure cannot be moved by a dynamic one either, thus generalizing and solidifying the first part of Wang and Davison's result. A follow-up paper discusses the second part.

I. Introduction

This paper is concerned with the stabilization of decentralized control systems, for which certain controller inputs may depend on some measurements but not others. This corresponds to finding a stabilizing controller which satisfies a given sparsity constraint. A special case of this, sometimes referred to as *perfectly decentralized control*, occurs when each control input may depend only on a single associated measurement, which corresponds to finding a stabilizing controller which is (block) diagonal.

This special case is sometimes itself referred to as *decentralized control*, particularly in the literature from a few decades ago. This malleability or evolution of the definition has not only caused some confusion, but has also resulted in some important results in the field only being studied for this special case.

We assume that plants and controllers are finite-dimensional, linear time-invariant (FDLTI), except for when we say otherwise, and will further assume that the plant is strictly proper.

A seminal result in decentralized control is the development of *fixed modes* by Wang and Davison in 1973 [1]. This paper studied (FDLTI) perfectly decentralized stabilization of FDLTI systems. Its contributions can be broken into three main components - a definition establishing the framework, and two subsequent results. Fixed modes were defined as those modes of the plant which could not be altered by any static perfectly decentralized controller (that is, by any diagonal matrix). The first result was that these fixed modes could also not be altered by any dynamic perfectly decentralized controller; if you can't move it with a static diagonal controller, you can't move it with a dynamic diagonal controller. The second result was that if a mode is not fixed, then it can be moved arbitrarily close to any chosen location in the complex plane (provided that it has a complex conjugate pair if it is not real). These can be taken together to state that a system is stabilizable by a (dynamic) perfectly decentralized controller if and only if all of its (static) fixed modes are in the left half-plane (LHP).

When proving these results, it was shown that allowing one part of the controller to be dynamic does not result in any fewer fixed modes than a static controller, and then claimed that the first result followed; that is, that a dynamic controller would not be able to move any of the fixed modes. Similarly, it was shown that a single non-fixed mode could be moved to any chosen location, and then claimed that the second result followed; that is, that an arbitrary number of non-fixed modes could be simultaneously moved to chosen locations by a single controller. Getting from these initial steps to a rigorous inductive argument, however, is not trivial.

We seek to study these fundamental concepts for arbitrary information structure, while developing robust notation and rigorous proofs, thus placing the new and existing results on a sound mathematical footing.

We first introduce notation for fixed modes that allows it to vary with information structure, as well as with the type of controllers allowed (static, dynamic, linear, etc.). We then show that, for arbitrary information structure, the fixed modes with respect to dynamic controllers are the same as the fixed modes with respect to static controllers, thus extending and solidifying part of the seminal results of Wang and Davison. In follow-up work [2], we study whether and how the non-fixed modes can be moved to chosen locations, extending and solidifying the remaining part of their work.

The obvious potential benefits of this are an increased understanding of decentralized stabilizability, and the verification of important existing results. It is also our hope that the notation developed will be useful in further extending our understanding of decentralized stabilizability to richer classes of controllers for which the fixed modes may diminish relative to the original static definition, particularly nonlinear and/or time-varying controllers [3]–[6]. We further note that demonstrating the results of this paper directly for

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arbitrary structure, as opposed to attempting to diagonalize the problem and then prove the original perfectly decentralized results, would likely be useful when other types of stability are required which are not invariant under such transformations, though we currently focus on internal state stability. As an example of the diagonalization approach, readers are referred to [7], where existence of a stabilizing controller under arbitrary information constraint has been demonstrated by transforming the problem into a diagonal one to which [1] could be applied. Furthermore, [7] demonstrates an analytical test for determining structural fixed modes under arbitrary information constraint and shows its equivalence to a graph-theoretical condition. We lastly note that dealing with the original structure is preferable in the subsequent work on sufficiency [2], as stabilizing controllers can be constructed without having to first expand their size.

The organization of this paper is as follows. In Section II we define notation and preliminaries, including our definition of fixed modes and the controller types that we will later need. In Section III we review certain previous results, specifically on controllability and observability with centralized controllers. In Section IV, we then state and prove our main results.

II. Preliminaries

We consider an FDLTI plant $P(\sigma)$ (where $\sigma = s, z$ depending on whether we are considering continuous or discrete cases, we use σ for statements that apply to both continuous time and discrete time cases). A state space representation of P is denoted by (A_P, B_P, C_P, D_P). All controllers under consideration in this paper will also be FDLTI. For an arbitrary controller K, we similarly denote a state-space representation by (A_K, B_K, C_K, D_K).

We will be imposing information constraints on our controller, which manifest as sparsity constraints. We suppose that we have a collection of pairs of indices (i, j) such that the ith controller input is allowed to depend on the jth measurement. We then define $\mathcal{S}$, our set of controllers with the desired structure, as those for which $K_{ij} = 0$ for any index pair which is not in this set. For a sparsity pattern $\mathcal{S}$, we similarly let $\mathrm{Adm}(\mathcal{S})$ denote the set of indices for which the controller is allowed to be non-zero, i.e., $(i, j) \notin \mathrm{Adm}(\mathcal{S})$ if and only if $K_{ij} = 0$ for all $K \in \mathcal{S}$. Also for simplicity we define the following sparsity patterns:

- $\mathcal{S}_{\rm c}$: Centralized sparsity patterns, i.e., no sparsity constraints are imposed on the controller. $\mathrm{Adm}(\mathcal{S}) = \{(i, j) \ \ \forall\, i, j\}$.
- $\mathcal{S}_{\rm d}$: Diagonal sparsity patterns, i.e., $K(\sigma)$ must be zero for all off-diagonal term (for almost all σ). $\mathrm{Adm}(\mathcal{S}) = \{(i, i) \ \ \forall\, i\}$.

We also define types of controllers that will help us to easily refer to whether a controller K is static, dynamic, or static for some elements but dynamic for others. We will make use of the following controller types:

- $\mathcal{T}^{\rm d}$: Set of finite order dynamic controllers, i.e., A_K, B_K, C_K, D_K each are real matrices of compatible dimension.
- $\mathcal{T}^{\rm s}$: Set of static controllers, i.e., A_K, B_K, and C_K are all zero and only D_K could be non-zero.
- $\mathcal{T}^{\rm s+1}_{i^+,j^+}$: Set of controllers such that all the elements of controller are static except for $(i^+, j^+)^{\rm th}$ element which could be dynamic, i.e., for all $(i, j) \neq (i^+, j^+)$, $K_{ij} \in \mathbb{R}$ and K_{i^+,j^+} is a proper transfer function in σ. This could be read as "static plus one".
- $\mathcal{T}^{\rm s+k}_{I}$: Set of controllers such that all the elements of controller are static except for k indices in the set $I \triangleq \{(i_1^+, j_1^+), \cdots, (i_k^+, j_k^+)\}$, i.e., for all $(i, j) \notin I$, $K_{ij} \in \mathbb{R}$ and for all $(i, j) \in I$, K_{ij} is a proper transfer function in σ. This could be read as "static plus k".

The closed-loop has a state-space representation with dynamics matrix denoted by $A_{\rm CL}(P, K)$, given by:

$$\begin{pmatrix} A_P + B_P M D_K C_P & B_P M C_K \\ B_K N C_P & A_K + B_K D_P M C_K \end{pmatrix}, \qquad (1)$$

where $M = (I - D_K D_P)^{-1}$, and $N = (I - D_P D_K)^{-1}$. Through the rest of this paper we assume that $D_P = 0$ and thus we have

$$A_{\rm CL}(P, K) = \begin{pmatrix} A_P + B_P D_K C_P & B_P C_K \\ B_K C_P & A_K \end{pmatrix}. \qquad (2)$$

Let $\mathbf{e}_k$ denote the unit vector of appropriate dimension, with all of its elements being zero, except for $k^{\rm th}$ element which is 1. Note that dimension of $\mathbf{e}_k$ should be clear from the context and thus we suppress the explicit dimension of $\mathbf{e}_k$ in the notation.

Definition 1: The set of fixed-modes of a plant P with respect to a sparsity pattern $\mathcal{S}$ and a type $\mathcal{T}$, is defined to be:

$$\Lambda\left(P, \mathcal{S}, \mathcal{T}\right) \triangleq \left\{\lambda \in \mathbb{C} \,|\, \lambda \in \mathrm{eig}\left(A_{\rm CL}(P, K)\right), \forall K \in \mathcal{S} \cap \mathcal{T}\right\} \qquad (3)$$

Remark 2: This reduces to the definition of fixed modes in [1] if $\mathcal{S} = \mathcal{S}_{\rm d}$ and $\mathcal{T} = \mathcal{T}^{\rm s}$.

III. Review

In this section we review controllability and observability of centralized linear time-invariant systems and their representation in Kalman canonical form with the help of the following lemma:

Lemma 3: For every plant P, there exists a similarity transformation matrix T such that

$$\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & D_P \end{array}\right] \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$
$$= \left[\begin{array}{cccc|c} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{B}_2 \\ 0 & 0 & \tilde{A}_{33} & 0 & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} & 0 \\ \hline \tilde{C}_1 & 0 & \tilde{C}_2 & 0 & D_P \end{array}\right]. \qquad (4)$$

In the above equation:

- $\tilde{A}_{11}$ corresponds to controllable and observable modes of P,

- $\tilde{A}_{22}$ corresponds to controllable and unobservable modes of P,
- $\tilde{A}_{33}$ corresponds to uncontrollable and observable modes of P,
- $\tilde{A}_{44}$ corresponds to uncontrollable and unobservable modes of P.

Proof: See, for example, [8]. ■

In order to reduce some of the notation, we do not explicitly show the dependence of $\tilde{A}_{ij}, \tilde{B}_i, \tilde{C}_j$ on A_P, B_P, C_P, and T, but it should be kept in mind that wherever we use Lemma 3 on a system, the resulting $\tilde{(\cdot)}$ variables are function of that system's state-space matrices with its respective Kalman similarity transformation matrix.

Next we state the following result regarding fixed modes with respect to a centralized sparsity pattern $\mathcal{S}_\text{c}$:

Lemma 4: For any plant P,

$$\Lambda\left(P, \mathcal{S}_\text{c}, \mathcal{T}^\text{s}\right) = \bigcup_{i=2,3,4} \text{eig}\left(\tilde{A}_{ii}\right), \tag{5}$$

where $\tilde{A}_{ii}$ are the blocks in the Kalman canonical decomposition of plant P, such that the fixed modes are the union of uncontrollable or unobservable modes of P.

Proof: See, for example, Theorem 2 in [9]. ■

Also to make this paper sufficiently self-contained we use our notation to restate the following result, which tells us that the fixed modes of a plant with centralized information structure are the same with respect to static or dynamic control:

Lemma 5: Given plant P,

$$\Lambda\left(P, \mathcal{S}_\text{c}, \mathcal{T}^\text{s}\right) = \Lambda\left(P, \mathcal{S}_\text{c}, \mathcal{T}^\text{d}\right). \tag{6}$$

Proof: See Lemma 3 of [1]. ■

IV. Main Result

We will show that for any arbitrary sparsity pattern $\mathcal{S}$, it still holds that the set of fixed modes with respect to static controllers is the same as the set of fixed modes with respect to dynamic controllers. We arrive at this result in Theorem 10, but first we will proceed with the following necessary steps.

We first state a lemma which is unsurprising but will be helpful. This lemma states that if λ is a fixed mode of a system with respect to static controllers and sparsity pattern $\mathcal{S}$, then after closing the loop with an arbitrary matrix $D_K \in \mathcal{S}$, if we further allow that only one of the static elements of the controller, namely $(i^+, j^+) \in \text{Adm}(\mathcal{S})$ to vary, then λ will still remain as a fixed mode. For ease of notation in the following Lemma and in rest of the paper, given any matrix $D_K \in \mathcal{S}$, and any $(i^+, j^+) \in \text{Adm}(\mathcal{S})$, define $P^+(\sigma)$, as illustrated in Figure 1, as:

$$P^+(\sigma) \triangleq \left[\begin{array}{c|c} A_P + B_P D_K C_P & B_{P,i^+} \\ \hline C_{P,j^+} & 0 \end{array}\right], \tag{7}$$

where $B_{P,i^+} \triangleq B_P \mathbf{e}_{i_+}$, and $C_{P,j^+} \triangleq \mathbf{e}_{j_+}^T C_P$, respectively denote i^+-th column of B_P, and j^+-th row of C_P.

Lemma 6: Given any matrix $D_K \in \mathcal{S}$, and for any $(i^+, j^+) \in \text{Adm}(\mathcal{S})$, if $\lambda \in \Lambda\left(P, \mathcal{S}, \mathcal{T}^\text{s}\right)$, then $\lambda \in \Lambda\left(P^+, \mathcal{S}_\text{c}, \mathcal{T}^\text{s}\right)$.

Proof: We use proof by contradiction. Assume $\lambda \in \Lambda\left(P, \mathcal{S}, \mathcal{T}^\text{s}\right)$, then by definition for any arbitrarily fixed $D_K \in \mathcal{S}$, we would have

$$\lambda \in \text{eig}\left(A_P + B_P D_K C_P\right). \tag{8}$$

Now suppose that $\lambda \notin \Lambda\left(P^+, \mathcal{S}_\text{c}, \mathcal{T}^\text{s}\right)$, then by definition there must exist a real matrix V such that

$$\lambda \notin \text{eig}\left(A_P + B_P D_K C_P + B_{P,i^+} V C_{P,j^+}\right). \tag{9}$$

Define the modified D_K as:

$$D_K^V \triangleq \begin{cases} D_K(i^+, j^+) + V, & \text{for } (i^+, j^+)^\text{th} \text{ element} \\ D_K(i,j), & \text{otherwise.} \end{cases}$$

Thus we have $B_P D_K^V C_P = B_P D_K C_P + B_{P,i^+} V C_{P,j^+}$. This, along with (9) would yield that $\lambda \notin \text{eig}\left(A_P + B_P D_K^V C_P\right)$, i.e., there exist a static controller $D_K^V \in \mathcal{S}$, such that $\lambda \notin \text{eig}\left(A_P + B_P D_K^V C_P\right)$, which shows $\lambda \notin \Lambda\left(P, \mathcal{S}, \mathcal{T}^\text{s}\right)$. This contradicts our first conjecture. ■

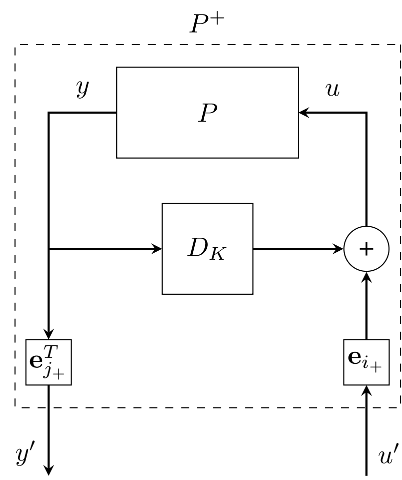

Fig. 1. P^+ is the SISO map from u' to y'.

Next, we relate fixed modes with respect to static controllers in $\mathcal{S}$ to the case where only one of the admissible elements of controller (in $\text{Adm}(\mathcal{S})$) is allowed to be dynamic, this result will be the foundation of the induction that we want to use later on. The outline of the proof is similar to that of [1, Proposition 1].

Theorem 7: For any sparsity pattern $\mathcal{S}$, and any arbitrarily fixed indices $(i^+, j^+) \in \text{Adm}(\mathcal{S})$,

$$\Lambda\left(P, \mathcal{S}, \mathcal{T}^\text{s}\right) = \Lambda\left(P, \mathcal{S}, \mathcal{T}^{\text{s}+1}_{i^+,j^+}\right). \tag{10}$$

Proof: $\supseteq$ follows immediately since $\mathcal{T}^\text{s} \subseteq \mathcal{T}^{\text{s}+1}_{i^+,j^+}$.

Now for the $\subseteq$ part, we first state the sparsity and type constraint of the RHS in terms of the controller's state-space representation and then exploit a special structure of the closed-loop A matrix to see the effect of the dynamic controller on the closed-loop eigenvalues.

Consider a controller $K \in \mathcal{S} \cap \mathcal{T}^{\mathrm{s}+1}_{i^+,j^+}$. This requires that $D_K \in \mathcal{S}$. If we denote the j^+-th column of B_K by B_{K,j^+}, and i^+-th row of C_K by C_{K,i^+}, then having $K \in \mathcal{T}^{\mathrm{s}+1}_{i^+,j^+}$ is equivalent to the existence of a state-space representation of K such that:

$$\begin{aligned} A_K &= \text{real matrix of appropriate size}, \\ B_K &= [0 \ \cdots 0 \ B_{K,j^+} \ 0 \cdots 0], \\ C_K &= [0 \ \cdots 0 \ C^T_{K,i^+} \ 0 \cdots 0]^T. \end{aligned} \tag{11}$$

Also as before we denote the i^+-th column of B_P, and the j^+-th row of C_P respectively by B_{P,i^+} and C_{P,j^+}.

Thus we can restate (3) in terms of constraints in (11) as:

$$\Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{s}+1}_{i^+,j^+}\right) = \bigcap_{\substack{D_K \in \mathcal{S} \\ \text{constraints in (11)}}} \mathrm{eig}\left(A_{\mathrm{CL}}(P,K)\right). \tag{12}$$

Now if we use (11) to replace for B_K, and C_K we have

$$A_{\mathrm{CL}}(P,K) = \begin{pmatrix} A_P + B_P D_K C_P & B_{P,i^+} C_{K,i^+} \\ B_{K,j^+} C_{P,j^+} & A_K \end{pmatrix}, \tag{13}$$

which simplifies (12) to

$$\Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{s}+1}_{i^+,j^+}\right) = \bigcap_{\substack{D_K \in \mathcal{S} \\ A_K, B_{K,j^+}, C_{K,i^+}}} \mathrm{eig}\left(A_{\mathrm{CL}}(P,K)\right). \tag{14}$$

If we now apply Lemma 3 on system P^+ to describe it in its Kalman canonical form, then using the corresponding transformation matrix T we can write:

$$\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|c} A_P + B_P D_K C_P & B_{P,i^+} \\ \hline C_{P,j^+} & 0 \end{array}\right] \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{cccc|c} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{B}_2 \\ 0 & 0 & \tilde{A}_{33} & 0 & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} & 0 \\ \hline \tilde{C}_1 & 0 & \tilde{C}_2 & 0 & 0 \end{array}\right], \tag{15}$$

which allows us to use a similarity transformation on close-loop A matrix to have the following special structure:

$$\mathrm{eig}\left(A_{\mathrm{CL}}(P,K)\right) = \mathrm{eig}\left(\begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} A_{\mathrm{CL}}(P,K) \begin{pmatrix} T^{-1} & 0 \\ 0 & I \end{pmatrix}\right)$$

$$\overset{(15)}{\underset{(13)}{=}} \mathrm{eig}\left(\begin{array}{cccc|c} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 & \tilde{B}_1 C_{K,i^+} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{B}_2 C_{K,i^+} \\ 0 & 0 & \tilde{A}_{33} & 0 & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} & 0 \\ \hline B_{K,j^+}\tilde{C}_1 & 0 & B_{K,j^+}\tilde{C}_2 & 0 & A_K \end{array}\right). \tag{16}$$

Let us introduce the temporary notation $\tilde{\mathbf{A}}$ to denote the big matrix in RHS of (16).

We need to show that if $\lambda \in \bigcup_{i=2,3,4} \mathrm{eig}\left(\tilde{A}_{ii}\right)$, then independent of A_K, B_{K,j^+}, and C_{K,i^+}, we have $\lambda \in \mathrm{eig}\left(\tilde{\mathbf{A}}\right)$. To see this notice that using an elementary similarity transformation $T' = \begin{bmatrix} \mathbf{e}_2 & \mathbf{e}_1 & \mathbf{e}_5 & \mathbf{e}_4 & \mathbf{e}_3 \end{bmatrix}$, results in an upper block triangular matrix with structure such that its eigenvalues clearly must include the eigenvalues of $\tilde{A}_{22}$, $\tilde{A}_{44}$, and $\tilde{A}_{33}$, and thus $\mathrm{eig}\left(\tilde{A}_{ii}\right) \subseteq \mathrm{eig}\left(\tilde{\mathbf{A}}\right)$, for $i = 2, 3, 4$.

We summarize and complete the proof by observing that:

$$\begin{aligned} \Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{s}}\right) &\overset{\text{Def 1}}{=} \bigcap_{D_K \in \mathcal{S}} \mathrm{eig}\left(A_P + B_P D_K C_P\right) \\ &\overset{\text{Lem 6}}{\subseteq} \bigcap_{D_K \in \mathcal{S}} \Lambda\left(\left[\begin{array}{c|c} A_P + B_P D_K C_P & B_{P,i^+} \\ \hline C_{P,j^+} & 0 \end{array}\right], \mathcal{S}_{\mathrm{c}}, \mathcal{T}^{\mathrm{s}}\right) \\ &\overset{\text{Lem 4}}{=} \bigcap_{D_K \in \mathcal{S}} \bigcup_{i=2,3,4} \mathrm{eig}\left(\tilde{A}_{ii}\right) \\ &\subseteq \bigcap_{D_K \in \mathcal{S}} \mathrm{eig}\left(\tilde{\mathbf{A}}\right), \quad \forall\, A_K, B_{K,j^+}, C_{K,i^+} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{s}}\right) &\subseteq \bigcap_{D_K \in \mathcal{S}, A_K, B_{K,j^+}, C_{K,i^+}} \mathrm{eig}\left(\tilde{\mathbf{A}}\right) \\ &\overset{(14)}{=} \Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{s}+1}_{i^+,j^+}\right). \end{aligned}$$

■

We note that it was this result, showing that modes which are fixed with respect to static controllers are still fixed with respect to "static plus one" controllers, that was established for $\mathcal{S} = \mathcal{S}_{\mathrm{d}}$ in [1], and at which point Theorem 10 was claimed to hold true. We will now show how to extend this result to show that modes which are fixed with respect to controllers with any given number of dynamic indices; that is, with respect to "static plus k" controllers, are still fixed when an additional index is allowed to become dynamic; that is, with respect to "static plus $k+1$" controllers. The main result will indeed follow once that has been established.

We will proceed with the following definitions. Let $K^{(k)}(\sigma)$ be the controller after k steps, with k of its indices allowed to be dynamic, and define $I^{(k)} \triangleq \left\{(i_1^+, j_1^+), \cdots, (i_k^+, j_k^+)\right\} \subset \mathrm{Adm}(\mathcal{S})$ as the set of such indices where $K^{(k)}(\sigma)$ is allowed to be dynamic, such that $K^{(k)} \in \mathcal{T}^{\mathrm{s}+k}_{I^{(k)}} \cap \mathcal{S}$. Also let $(A_K^{(k)}, B_K^{(k)}, C_K^{(k)}, D_K^{(k)})$ be a state-space representation for $K^{(k)}(\sigma)$.

Define $P^{(k)}(\sigma)$, illustrated in Figure 2, by closing $K^{(k)}(\sigma)$ around $P(\sigma)$ in such a way that the outputs of $P^{(k)}$ are the same as the outputs of P, and such that the inputs of $P^{(k)}$ are added to the outputs of $K^{(k)}$ and fed into P.

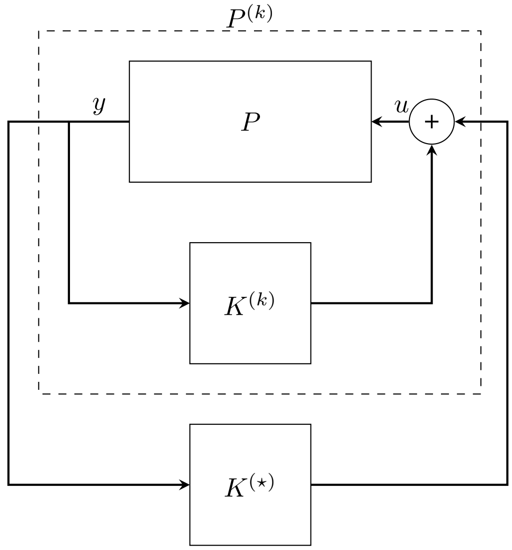

Fig. 2. Plant $P^{(k)}$ and its respective controller $K^{(\star)}$.

A state-space representation for $P^{(k)}(\sigma)$ is:

$$\begin{aligned} P^{(k)}(\sigma) &= \left[\begin{array}{c|c} A_{\mathrm{CL}}(P, K^{(k)}) & \begin{matrix} B_P \\ 0 \end{matrix} \\ \hline \begin{matrix} C_P & 0 \end{matrix} & D_P \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_P + B_P D_K^{(k)} C_P & B_P C_K^{(k)} & B_P \\ B_K^{(k)} C_P & A_K^{(k)} & 0 \\ \hline C_P & 0 & D_P \end{array}\right]. \end{aligned} \tag{17}$$

We prove one remaining lemma before our main inductive step. This lemma relates the modes which are fixed when closing controllers with $k+1$ dynamic elements around the plant, to the modes which are fixed when first closing controllers with k dynamic elements around the plant, and then closing a controller with an additional dynamic element around the resulting plant, as in Figure 2. This will allow us to use our result relating static and "static plus one" controllers to make conclusions relating "static plus k" an "static plus $k+1$" controllers.

Lemma 8: Given a set of indices $I^{(k)} \subset \mathrm{Adm}(\mathcal{S})$, an additional index pair $(i^+_{k+1}, j^+_{k+1}) \in \mathrm{Adm}(\mathcal{S}) \setminus I^{(k)}$, and $I^{(k+1)} = I^{(k)} \cup (i^+_{k+1}, j^+_{k+1})$, with $P^{(k)}$ defined as (17), we have

$$\Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{s}+k+1}_{I^{(k+1)}}\right) = \bigcap_{K^{(k)} \in \mathcal{S} \cap \mathcal{T}^{\mathrm{s}+k}_{I^{(k)}}} \Lambda\left(P^{(k)}, \mathcal{S}, \mathcal{T}^{\mathrm{s}+1}_{i^+_{k+1}, j^+_{k+1}}\right). \tag{18}$$

Proof: For ease of notation, when the controllers are unambiguous such that we can suppress the dependency upon them, define $A^{\mathrm{LHS}}_{\mathrm{CL}} = A_{\mathrm{CL}}(P, K^{(k+1)})$, and $A^{\mathrm{RHS}}_{\mathrm{CL}} = A_{\mathrm{CL}}(P^{(k)}, K^{(\star)})$, the closed-loop dynamics matrices arising on each side of the equation for given controllers. Also let $\mathcal{K}_{\mathrm{LHS}} \triangleq \{K^{(k+1)} | K^{(k+1)} \in \mathcal{T}^{\mathrm{s}+k+1}_{I^{(k+1)}} \cap \mathcal{S}\}$, and $\mathcal{K}_{\mathrm{RHS}} \triangleq \{(K^{(k)}, K^{(\star)}) | K^{(k)} \in \mathcal{T}^{\mathrm{s}+k}_{I^{(k)}} \cap \mathcal{S}, K^{(\star)} \in \mathcal{T}^{\mathrm{s}+1}_{i^+_{k+1}, j^+_{k+1}} \cap \mathcal{S}\}$ give the sets of controllers that must be considered on each side, such that the LHS can be abbreviated as $\bigcap_{\mathcal{K}_{\mathrm{LHS}}} \mathrm{eig}\left(A^{\mathrm{LHS}}_{\mathrm{CL}}\right)$, and the RHS can be abbreviated as $\bigcap_{\mathcal{K}_{\mathrm{RHS}}} \mathrm{eig}\left(A^{\mathrm{RHS}}_{\mathrm{CL}}\right)$.

First we prove the $\subseteq$ part by showing that for every admissible $K^{(\star)}(\sigma)$, i.e., $K^{(\star)} \in \mathcal{S} \cap \mathcal{T}^{\mathrm{s}+1}_{i^+_{k+1}, j^+_{k+1}}$, and admissible $K^{(k)}(\sigma)$ in RHS, there exist a $K^{(k+1)}(\sigma)$ in LHS such that $A^{\mathrm{RHS}}_{\mathrm{CL}} = A^{\mathrm{LHS}}_{\mathrm{CL}}$:

$$\begin{aligned} &A^{\mathrm{RHS}}_{\mathrm{CL}} \\ &= \begin{pmatrix} A_{P^{(k)}} + B_{P^{(k)}} D_K^{(\star)} C_{P^{(k)}} & B_{P^{(k)}} C_K^{(\star)} \\ B_K^{(\star)} C_{P^{(k)}} & A_K^{(\star)} \end{pmatrix} \\ &= \begin{pmatrix} A_P + B_P \left(D_K^{(k)} + D_K^{(\star)}\right) C_P & B_P C_K^{(k)} & B_P C_K^{(\star)} \\ B_K^{(k)} C_P & A_K^{(k)} & 0 \\ B_K^{(\star)} C_P & 0 & A_K^{(\star)} \end{pmatrix} \\ &= \begin{pmatrix} A_P + B_P \left(D_K^{(k)} + D_K^{(\star)}\right) C_P & B_P \begin{bmatrix} C_K^{(k)} & C_K^{(\star)} \end{bmatrix} \\ \begin{bmatrix} B_K^{(k)} \\ B_K^{(\star)} \end{bmatrix} C_P & \begin{bmatrix} A_K^{(k)} & 0 \\ 0 & A_K^{(\star)} \end{bmatrix} \end{pmatrix} \\ &= \begin{pmatrix} A_P + B_P D_K^{(k+1)} C_P & B_P C_K^{(k+1)} \\ B_K^{(k+1)} C_P & A_K^{(k+1)} \end{pmatrix} \\ &= A^{\mathrm{LHS}}_{\mathrm{CL}}, \end{aligned} \tag{19}$$

where we have chosen $K^{(k+1)}(\sigma) = K^{(k)}(\sigma) + K^{(\star)}(\sigma)$. This $K^{(k+1)}$ is admissible because it has only one further dynamic element at position $(i^+_{k+1}, j^+_{k+1}) \in \mathrm{Adm}(\mathcal{S})$, and thus is in $\mathcal{T}^{\mathrm{s}+k+1}_{I^{(k+1)}}$. Hence for every admissible $(K^{(K)}, K^{(\star)})$, there exists an admissible $K^{(k+1)} \in \mathcal{K}_{\mathrm{LHS}}$ constructed as above such that $A^{\mathrm{LHS}}_{\mathrm{CL}} = A^{\mathrm{RHS}}_{\mathrm{CL}}$, and so $\bigcap_{\mathcal{K}_{\mathrm{LHS}}} \mathrm{eig}\left(A^{\mathrm{LHS}}_{\mathrm{CL}}\right) \subseteq \bigcap_{\mathcal{K}_{\mathrm{RHS}}} \mathrm{eig}\left(A^{\mathrm{RHS}}_{\mathrm{CL}}\right)$.

We will prove the $\supseteq$ part by contradiction. Assume that this does not hold, and thus that there exists a λ such that $\lambda \in \bigcap_{K^{(k)} \in \mathcal{S} \cap \mathcal{T}^{\mathrm{s}+k}_{I^{(k)}}} \Lambda\left(P^{(k)}, \mathcal{S}, \mathcal{T}^{\mathrm{s}+1}_{i^+_{k+1}, j^+_{k+1}}\right)$, but $\lambda \notin \Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{s}+k+1}_{I^{(k+1)}}\right)$. Then we have:

$$\forall \left(K^{(k)}, K^{(\star)}\right) \in \mathcal{K}_{\mathrm{RHS}}, \ \lambda \in \mathrm{eig}\left(A_{\mathrm{CL}}(P^{(k)}, K^{(\star)})\right) \tag{20a}$$

$$\exists\, K^{(k+1)} \in \mathcal{K}_{\mathrm{LHS}} \ \text{ s.t. } \ \lambda \notin \mathrm{eig}\left(A_{\mathrm{CL}}(P, K^{(k+1)})\right). \tag{20b}$$

Starting with $K^{(k+1)}$ from (20b), we will show that we can then construct a $K^{(k)}$ and $K^{(\star)}$ to falsify (20a).

Based on $K^{(k+1)}$ in (20b), we let $\tilde{K}^{(\star)}$ be the dynamic part of the final dynamic index by defining $\tilde{K}^{(\star)} =$

$(\tilde{A_K}^{(\star)}, \tilde{B_K}^{(\star)}, \tilde{C_K}^{(\star)}, \tilde{D_K}^{(\star)})$ as:

$$\begin{aligned} \tilde{A_K}^{(\star)} &= A_K^{(k+1)}, \\ \tilde{B_K}^{(\star)} &= \begin{bmatrix} 0 & \cdots & B_{K,j_{k+1}^+}^{(k+1)} & \cdots & 0 \end{bmatrix}, \\ \tilde{C_K}^{(\star)} &= \begin{bmatrix} 0 & \cdots & (C_{K,i_{k+1}^+}^{(k+1)})^T & \cdots & 0 \end{bmatrix}^T, \\ \tilde{D_K}^{(\star)} &= 0, \end{aligned} \tag{21}$$

i.e., $\tilde{B_K}^{(\star)}$ is of the same dimension as $B_K^{(k+1)}$ with all its columns being zero except j_{k+1}^+ column, and $\tilde{C_K}^{(\star)}$ is of the same dimension as $C_K^{(k+1)}$ with all of its rows being zero except i_{k+1}^+ row. Then define $\tilde{K}^{(k)} \triangleq K^{(k+1)} - \tilde{K}^{(\star)}$, thus a state-space representation for $\tilde{K}^{(k)}$ is:

$$\begin{aligned} \tilde{A_K}^{(k)} &= \mathrm{diag}(A_K^{(k+1)}, A_K^{(k+1)}), \\ \tilde{B_K}^{(k)} &= \begin{bmatrix} (B_K^{(k+1)})^T & (\tilde{B_K}^{(\star)})^T \end{bmatrix}^T, \\ \tilde{C_K}^{(k)} &= \begin{bmatrix} C_K^{(k+1)} & -\tilde{C_K}^{(\star)} \end{bmatrix}, \\ \tilde{D_K}^{(k)} &= D_K^{(k+1)}. \end{aligned} \tag{22}$$

Construct $\tilde{P}^{(k)}$ in the same way as illustrated in Figure 2 by closing $\tilde{K}^{(k)}$ around P. Now if we use the following similarity transformation T on $A_{\mathrm{CL}}(\tilde{P}^{(k)}, \tilde{K}^{(\star)})$,

$$T = \begin{bmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & I \end{bmatrix},$$

then $TA_{\mathrm{CL}}(\tilde{P}^{(k)}, \tilde{K}^{(\star)})T^{-1}$ results in an upper block triangular matrix with blocks $A_K^{(k+1)}$, $A_{\mathrm{CL}}(P, K^{(k+1)})$, and $A_K^{(k+1)}$, indicating that:

$$\begin{aligned} &\mathrm{eig}\left(A_{\mathrm{CL}}(\tilde{P}^{(k)}, \tilde{K}^{(\star)})\right) = \\ &\qquad \mathrm{eig}\left(A_{\mathrm{CL}}(P, K^{(k+1)})\right) \cup \mathrm{eig}\left(A_K^{(k+1)}\right). \end{aligned} \tag{23}$$

Thus (20) and (23) imply that for that $K^{(k+1)}(\sigma)$, we necessarily have

$$\lambda \in \mathrm{eig}\left(A_K^{(k+1)}\right). \tag{24}$$

We have shown that the only way to have an eigenvalue which is not on the LHS (when $K^{(k+1)}$ is closed around the plant) but which is on the RHS (when $\tilde{K}^{(\star)}$ and $\tilde{K}^{(k)}$ are then constructed as above), is if it comes from the dynamics matrix of $K^{(k+1)}$. We will now finish the proof by showing that if this is the case, we can make a small perturbation to the matrix such that it no longer has this eigenvalue, thus removing it from the RHS, while it is still not a closed-loop eigenvalue on the LHS.

Perturb $\tilde{K}^{(\star)}$, $\tilde{K}^{(k)}$, and $K^{(k+1)}$ by adding ϵI to their A matrices and name them respectively as $\hat{K}^{(\star)}$, $\hat{K}^{(k)}$, and $\hat{K}^{(k+1)}$, for example $\hat{K}^{(\star)}$ would be defined as:

$$\begin{aligned} \hat{A_K}^{(\star)} &= \tilde{A_K}^{(\star)} + \epsilon I, & \hat{B_K}^{(\star)} &= \tilde{B_K}^{(\star)}, \\ \hat{C_K}^{(\star)} &= \tilde{C_K}^{(\star)}, & \hat{D_K}^{(\star)} &= \tilde{D_K}^{(\star)}. \end{aligned}$$

These perturbations for sufficiently small ϵ, along with (24) yield that

$$\lambda \notin \mathrm{eig}\left(\hat{A_K}^{(\star)}\right), \lambda \notin \mathrm{eig}\left(\hat{A_K}^{(k)}\right), \ \lambda \notin \mathrm{eig}\left(\hat{A_K}^{(k+1)}\right). \tag{25}$$

Using the same similarity transformation T, and exactly similar as deriving (23), we have

$$\begin{aligned} &\mathrm{eig}\left(A_{\mathrm{CL}}(\hat{P}^{(k)}, \hat{K}^{(\star)})\right) = \\ &\qquad \mathrm{eig}\left(A_{\mathrm{CL}}(P, \hat{K}^{(k+1)})\right) \cup \mathrm{eig}\left(\hat{A_K}^{(k+1)}\right), \end{aligned} \tag{26}$$

where $\hat{P}^{(k)}$ is constructed, as illustrated in Figure 2, by closing $\hat{K}^{(k)}$ around P.

Since $A_{\mathrm{CL}}(P, K^{(k+1)})$ is continuous in the entries of $K^{(k+1)}$, and since the eigenvalues of a matrix are continuous in its entries (c.f. [10, Theorem 5.2. on p. 89]), it follows from the sufficiently small perturbation made to $K^{(k+1)}$ and from (20b) that we still have $\lambda \notin \mathrm{eig}\left(A_{\mathrm{CL}}(P, \hat{K}^{(k+1)})\right)$. It then follows from (25) and (26) that $\lambda \notin \mathrm{eig}\left(A_{\mathrm{CL}}(\hat{P}^{(k)}, \hat{K}^{(\star)})\right)$.

Thus we have been able to show that there exists a $(\hat{K}^{(k)}, \hat{K}^{(\star)}) \in \mathcal{K}_{\mathrm{RHS}}$ such that $\lambda \notin \mathrm{eig}\left(A_{\mathrm{CL}}(\hat{P}^{(k)}, \hat{K}^{(\star)})\right)$, which contradicts our assumption.

∎

Now we are ready to prove our main inductive step: that given a certain number of controller indices which are allowed to be dynamic, and the associated set of fixed modes, allowing one additional index to become dynamic does not change the fixed modes.

Theorem 9: Given any plant P, a sparsity pattern $\mathcal{S}$, an admissible set of dynamic elements at step k denoted by $I^{(k)} \subset \mathrm{Adm}(\mathcal{S})$, an index pair $(i_{k+1}^+, j_{k+1}^+) \in \mathrm{Adm}(\mathcal{S}) \setminus I^{(k)}$ that is further allowed to be dynamic at step $k+1$, and the resulting $I^{(k+1)} = I^{(k)} \cup (i_{k+1}^+, j_{k+1}^+)$, we have:

$$\Lambda\left(P, \mathcal{S}, \mathcal{T}_{I^{(k)}}^{\mathrm{s}+k}\right) = \Lambda\left(P, \mathcal{S}, \mathcal{T}_{I^{(k+1)}}^{\mathrm{s}+k+1}\right). \tag{27}$$

Proof:

$$\begin{aligned} \mathrm{RHS} &\overset{\text{Lem.8}}{=} \bigcap_{K^{(k)} \in \mathcal{S} \cap \mathcal{T}_{I^{(k)}}^{\mathrm{s}+k}} \Lambda\left(P^{(k)}, \mathcal{S}, \mathcal{T}_{i_{k+1}^+, j_{k+1}^+}^{\mathrm{s}+1}\right) \\ &\overset{\text{Thm.7}}{=} \bigcap_{K^{(k)} \in \mathcal{S} \cap \mathcal{T}_{I^{(k)}}^{\mathrm{s}+k}} \Lambda\left(P^{(k)}, \mathcal{S}, \mathcal{T}^{\mathrm{s}}\right) \\ &= \Lambda\left(P, \mathcal{S}, \mathcal{T}_{I^{(k)}}^{\mathrm{s}+k}\right) = \mathrm{LHS} \end{aligned}$$

∎

We can now state and easily prove our main result. The following shows that for any FDLTI plant P, and any sparsity pattern $\mathcal{S}$, the set of fixed modes with respect to static and dynamic controllers are the same.

Theorem 10: Given plant P, and sparsity constraint $\mathcal{S}$:

$$\Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{s}}\right) = \Lambda\left(P, \mathcal{S}, \mathcal{T}^{\mathrm{d}}\right). \tag{28}$$

Proof: This follows by induction from Theorem 9. ■

V. Conclusion

We revisited, verified, and generalized classic work in the stabilizability of decentralized systems. We generalized the notion of fixed modes to arbitrary information structure, and provided a rigorous inductive proof that plant modes which cannot be moved by static controllers with the prescribed structure cannot be moved by dynamic controllers either. Work to similarly solidify and generalize the other main result of [1], dealing with the placement of the modes which are not fixed, appears in a follow-up paper [2].

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