

Q-Parametrization and an SDP for $\mathcal{H}_\infty$ -optimal Decentralized Control

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Abstract: We consider the problem of finding decentralized controllers to optimize an $\mathcal{H}_\infty$ -norm. This can be cast as a convex optimization problem when certain conditions are satisfied, but it is an infinite-dimensional problem that still cannot be addressed with existing methods. We use Q -parametrization to approach the original problem with a sequence of finite-dimensional problems. A method is discussed to solve the resulting finite-dimensional approximated convex problem. It is then shown how this problem can be cast as a semidefinite program and generally solved much more efficiently.

Keywords: Decentralized control, networked control, $\mathcal{H}_\infty$ -control, convex optimization, semidefinite programming, quadratic invariance

1. INTRODUCTION

Decentralized control has long been the focus of a variety of research, as it is of great interest to have controllers which rely only on local measurements in a complex system. The design of decentralized controllers with optimal performance is intractable in general however.

The main techniques for centralized $\mathcal{H}_\infty$ control, linear matrix inequalities (LMIs) (Gahinet and Apkarian (1994)) and Ricatti equations (Doyle et al. (1989)), do not allow one to optimize directly over the controller, and thus do not present an obvious way to allow one to place constraints on the controller. There has thus been a variety of methods trying to solve for structured $\mathcal{H}_\infty$ controller by approximation, including homotopy methods for the non-convex bilinear matrix inequality, cf. (Zhai et al. (2001)), finding local optima with non-smooth optimization techniques, cf. (Bompart et al. (2007)), and an approach based on dissipative property of systems which result in sub-optimal $\mathcal{H}_\infty$ controller design by LMIs, cf. (Scorletti and Duc (1997)). There are relationships between structured control problems and multi-objective problems (Scherer (2002)), and a finite-dimensional basis parametrization has been used to approach the solution for the latter (Hindi et al. (1998)). Subsequent to the initial submission of this paper was an interesting development (Scherer (2013)), whereby if both the plant and controller admit lower triangular structure, exact $\mathcal{H}_\infty$ -optimal state-space solutions may be found.

When the problem one wishes to solve satisfies a certain condition, called quadratic invariance, the optimal decentralized control problem may be cast as a convex optimization problem, regardless of which closed-loop norm the designer wishes to optimize (Rotkowitz and Lall (2006b)).

The resulting problem is infinite-dimensional and still non-trivial, particularly for certain objectives. When the objective is the $\mathcal{H}_2$ -norm, it was shown in (Rotkowitz and Lall (2006b)) that the problem can be further reduced to an unconstrained optimal control problem and then solved with standard software. Some recent progress has also been made to directly compute the optimal state-space controller parameters for some specific information constraints in the $\mathcal{H}_2$ case (Lamperski and Doyle (2012); Lessard (2012); Shah and Parrilo (2010)).

As with centralized control, there are many cases where one must optimize for the worst-case, such as the decentralized control of smart structures to prevent failure during earthquakes (Wang et al. (2009)), and thus for which $\mathcal{H}_\infty$ -norm is more appropriate objective. There are further well-known advantages in optimizing the $\mathcal{H}_\infty$ -norm with regards to robustness (Zames (1981)). We thus address optimal decentralized control for the $\mathcal{H}_\infty$ -norm in this paper.

When the problem is quadratically invariant, we use Q -parametrization to get a sequence of finite-dimensional problems whose solutions approach that of our original problem, and discuss methods for solving these, which could easily be adapted for any norm of interest. For our main result, we show how the main result of (Scherer (2000)) can be used to recast any of the optimization problems in this sequence as a semidefinite program (SDP) when the norm of interest is indeed the $\mathcal{H}_\infty$ -norm.

Although this paper mainly focuses on discrete-time systems, by selection of appropriate basis, continuous time counter parts could be derived similarly. The paper also focuses on sparsity constraints, where each control action may be a function of some measurements but not others,

but one could easily adapt the developed methods for delay constraints, where each control action may also be a function of certain measurements after a specified amount of time has passed.

This paper is organized as follows. Section 2 will introduce some notations and preliminary definitions used through this paper. We formulate the problem of finding an optimal decentralized controller in Section 3. Quadratic Invariance (QI) is discussed in Section 4. Section 5 demonstrates the finite basis used for approximating an infinite dimensional controller and its corresponding state-space representation. In Section 6, we show how, for any norm, Q -parametrized decentralized controller could be thought of as static output feedback (SOF) with sparsity pattern imposed on a static gain (whenever quadratic invariance is satisfied), and proceed to solve it in case of $\mathcal{H}_\infty$ norm with help of method proposed by (Scherer (2000)). A numerical example is provided in Section 7 to further inspect different aspects of methods discussed throughout this paper.

2. PRELIMINARIES

We define some standard notation, mainly for the function spaces needed, in Section 2.1, develop the two-input two-output framework in Section 2.2, and introduce some notation that will help encapsulate the main type of decentralization we consider in Section 2.3.

2.1 Function spaces

We use the following standard notation. Denote the unit disk by $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and unit circle by $\partial\mathbb{D}$, and the closed space outside the unit disk by $\mathbb{D}_+ = \{z \in \mathbb{C} \mid |z| \geq 1\}$. We define transfer functions for discrete-time systems determined on the unit circle. A rational function $G : \partial\mathbb{D} \rightarrow \mathbb{C}$ is called **real-rational** if the coefficients of its numerator and denominator polynomials are real. Similarly, a matrix-valued function $G : \partial\mathbb{D} \rightarrow \mathbb{C}^{m \times n}$ is called real-rational if G_{ij} is real-rational for all i, j . Denote by $\mathcal{R}_p^{m \times n}$ the set of matrix-valued real-rational proper transfer matrices $\mathcal{R}_p^{m \times n} = \{G : \partial\mathbb{D} \rightarrow \mathbb{C}^{m \times n} \mid G \text{ proper, real-rational}\}$ and let $\mathcal{R}_{sp}^{m \times n}$ be $\mathcal{R}_{sp}^{m \times n} = \{G \in \mathcal{R}_p^{m \times n} \mid G \text{ strictly proper}\}$. Also let $\mathcal{RH}_\infty$ be the set of real-rational proper stable transfer matrices $\mathcal{RH}_\infty^{m \times n} = \{G \in \mathcal{R}_p^{m \times n} \mid G \text{ has no poles in } \mathbb{D}_+\}$. It can be shown that functions in $\mathcal{RH}_\infty$ are determined by their values on $\partial\mathbb{D}$, and thus we can regard $\mathcal{RH}_\infty$ as a subspace of $\mathcal{R}_p$.

Unless otherwise declared, vector norms in this paper are all standard Euclidean norm $\|v\|^2 = v^*v = \bar{v}^T v$ for $v \in \mathbb{C}^n$ where superscript $*$ is Hermitian, and inner product is the standard dot product $\langle u, v \rangle = u^*v$.

A transfer function matrix $G \in \mathcal{H}_\infty$ iff G is analytic on $\mathbb{D}_+$ and $\text{ess sup}_{\omega \in [0, 2\pi)} \sigma_{\max}(G(e^{j\omega})) < \infty$, where $\sigma_{\max}(\cdot)$ gives the maximum singular value, and for m -by- n $G \in \mathcal{H}_\infty$ its norm is given by

$$\begin{aligned} \|G\|_{\mathcal{H}_\infty} &= \text{ess sup}_{\omega \in [0, 2\pi)} \sigma_{\max}(G(e^{j\omega})) \\ &= \text{ess sup} \Re(u^* G(e^{j\omega})v) \end{aligned}$$

where $\Re(\cdot)$ gives the real part of a complex number, and the last essential supremum is taken over $\omega \in [0, 2\pi)$, $u \in \mathbb{C}^m$, $v \in \mathbb{C}^n$ with $\|u\| = \|v\| = 1$.

We could now equivalently define $\mathcal{RH}_\infty^{m \times n} = \mathcal{R}_p^{m \times n} \cap \mathcal{H}_\infty^{m \times n}$. When the dimensions are implied by context, we omit the superscripts of $\mathcal{R}_p^{m \times n}$, $\mathcal{R}_{sp}^{m \times n}$, $\mathcal{RH}_\infty^{m \times n}$, $\mathcal{H}_\infty^{m \times n}$. Let I_n represent the $n \times n$ identity.

2.2 Plant and controller

We suppose that we have a causal, linear time-invariant, discrete-time generalized plant $P \in \mathcal{R}_p^{(n_z+n_y) \times (n_w+n_u)}$, partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & G \end{bmatrix}.$$

Given a controller $K \in \mathcal{R}_p^{n_u \times n_y}$, we define the **closed-loop map** by $f(P, K) \stackrel{\text{def}}{=} P_{11} + P_{12}K(I - GK)^{-1}P_{21}$. The map $f(P, K)$ is also called the (lower) **linear fractional transformation** (LFT) of P and K . Note that we abbreviate $G = P_{22}$, since we will refer to that block frequently, and so that we may refer to its subdivisions without ambiguity. This interconnection is shown in Figure 1.

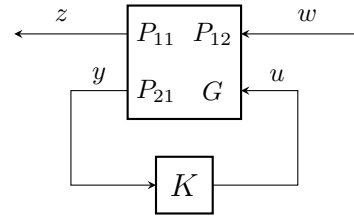


Fig. 1. Linear fractional interconnection of P and K

We suppose that there are n_y sensor measurements and n_u control actions, and thus partition the sensor measurements and control actions as

$$y = [y_1^T \dots y_{n_y}^T]^T \quad u = [u_1^T \dots u_{n_u}^T]^T$$

and then further partition G and K as

$$G = \begin{bmatrix} G_{11} & \dots & G_{1n_u} \\ \vdots & & \vdots \\ G_{n_y1} & \dots & G_{n_y n_u} \end{bmatrix} \quad K = \begin{bmatrix} K_{11} & \dots & K_{1n_y} \\ \vdots & & \vdots \\ K_{n_u1} & \dots & K_{n_u n_y} \end{bmatrix}$$

This will typically represent n subsystems, each with its own controller, in which case we will have $n = n_y = n_u$, but this does not have to be the case.

2.3 Sparsity patterns

Let $\mathbb{B} = \{0, 1\}$ represent the set of binary numbers.

Suppose that $A^{\text{bin}} \in \mathbb{B}^{m \times n}$ is a binary matrix. The following is the subspace of $\mathcal{R}_p^{m \times n}$ comprising the transfer function matrices that satisfy the sparsity constraints imposed by A^{bin} :

$$\begin{aligned} \text{Sparse}(A^{\text{bin}}) &\stackrel{\text{def}}{=} \{B \in \mathcal{R}_p^{m \times n} \mid B_{ij}(e^{j\omega}) = 0 \text{ for all } i, j \\ &\text{s.t. } A_{ij}^{\text{bin}} = 0 \text{ for almost all } \omega \in [0, 2\pi]\} \end{aligned}$$

Conversely, given $B \in \mathcal{R}_p^{m \times n}$, we define $\text{Pattern}(B) \stackrel{\text{def}}{=} A^{\text{bin}}$, where A^{bin} is the binary matrix given by:

$$A_{ij}^{\text{bin}} = \begin{cases} 0, & \text{if } B_{ij}(e^{j\omega}) = 0 \text{ for almost all } \omega \in [0, 2\pi] \\ 1, & \text{otherwise} \end{cases},$$

for $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$.

Remark 1. Note that one may often wish to generalize these definitions to account for a partitioning of the transfer function matrix, with the binary matrix instead capturing which blocks are non-zero, but as the respective generalizations of the results are obvious, we will not bother with the additional notation at this time.

3. PROBLEM FORMULATION

We now introduce the main class of problem we will consider in this paper, and then formulate our optimization problem.

3.1 Sparsity Constraints

We consider sparsity constraints on controller such that each control input may access certain sensor measurements, but not others.

We represent sparsity constraints on the overall controller via a binary matrix $K^{\text{bin}} \in \mathbb{B}^{n_u \times n_y}$. Its entries can be interpreted as follows:

$$K_{kl}^{\text{bin}} = \begin{cases} 1, & \text{if control input } k \\ & \text{may access sensor measurement } l, \\ 0, & \text{if not,} \end{cases}$$

for all $k \in \{1, \dots, n_u\}$, $l \in \{1, \dots, n_y\}$.

The subspace of admissible controllers can be expressed as:

$$\mathcal{S} = \text{Sparse}(K^{\text{bin}}).$$

The sparsity pattern of the plant, which is especially relevant to the controller optimization, is obtained as:

$$G^{\text{bin}} = \text{Pattern}(G)$$

where G^{bin} is interpreted as follows:

$$G_{ij}^{\text{bin}} = \begin{cases} 1, & \text{if control input } j \\ & \text{affects sensor measurement } i, \\ 0, & \text{if not,} \end{cases}$$

for all $i \in \{1, \dots, n_y\}$, $j \in \{1, \dots, n_u\}$.

3.2 Problem Setup

Given a generalized plant P and a subspace of admissible controllers $\mathcal{S}$, the following is the optimal decentralized control problem we seek to address:

$$\begin{aligned} & \underset{K \in \mathcal{R}_p}{\text{minimize}} && \|f(P, K)\|_{\mathcal{H}_\infty} \\ & \text{subject to} && K \text{ stabilizes } P \\ & && K \in \mathcal{S} \end{aligned} \quad (1)$$

Which subsystems can affect others is embedded in the sparsity pattern of P , and which subsystem controllers can access the sensor information from which others is embedded in $\mathcal{S}$. We call the subspace $\mathcal{S}$ the **information constraint**.

Many decentralized control problems may be expressed in the form of problem (1), including all of those addressed in (Qi et al. (2004); Siljak (1994)).

This problem is made substantially more difficult in general by the constraint that K lie in the subspace $\mathcal{S}$. Without this constraint, the problem could be solved with

many standard techniques. Note that the cost function $\|f(P, K)\|$ is in general a non-convex function of K . No computationally tractable approach is known for solving this problem for arbitrary P and $\mathcal{S}$. For some P and $\mathcal{S}$, the problem has been shown to be equivalent to a convex optimization problem.

This is the subject of the next section, and we will then focus on methods to solve those problems for the $\mathcal{H}_\infty$ -norm.

4. QUADRATIC INVARIANCE

In this section, we define quadratic invariance, and we give a brief overview of related results, in particular, that if it holds then convex synthesis of optimal decentralized controllers is possible. We then discuss what it means for the class of problems that we focus on in this paper, control subject to sparsity constraints.

Definition 2. Let a causal linear time-invariant plant, represented via a transfer function matrix G in $\mathcal{R}_p^{n_y \times n_u}$, be given. If $\mathcal{S}$ is a subset of $\mathcal{R}_p^{n_u \times n_y}$ then $\mathcal{S}$ is called **quadratically invariant** under G if the following inclusion holds:

$$K G K \in \mathcal{S} \quad \text{for all } K \in \mathcal{S}.$$

It was shown in (Rotkowitz and Lall (2006b)) that if $\mathcal{S}$ is a closed subspace and $\mathcal{S}$ is quadratically invariant under G , then with a change of variables, problem (1) is equivalent to the following optimization problem:

$$\begin{aligned} & \underset{Q \in \mathcal{RH}_\infty}{\text{minimize}} && \|T_1 - T_2 Q T_3\|_{\mathcal{H}_\infty} \\ & \text{subject to} && Q \in \mathcal{S} \end{aligned} \quad (2)$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$. See Theorem 17 in (Rotkowitz and Lall (2006b)) for finding T_1, T_2, T_3 and recovering K from Q . Throughout the rest of the paper we will focus on this equivalent form instead of (1).

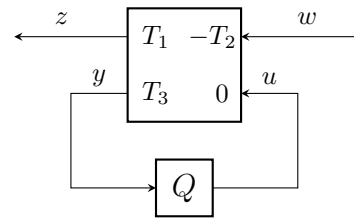


Fig. 2. Model-matching problem from Youla parametrization

This states that if our problem is quadratically invariant (QI), we may use a particular Youla parametrization (Youla et al. (1976)) to reduce the problem to the model-matching problem shown in Figure 2, as one can for centralized problems, and the constraint on the controller is passed on to the Youla parameter. The optimization problem in (2) is then convex. We may solve it to find the optimal Q , and then recover the optimal K for our original problem (1). Similar results have been achieved (Rotkowitz and Lall (2006a)) for other function spaces as well, also showing that quadratic invariance

allows optimal linear decentralized control problems to be recast as convex optimization problems. This result also holds, and most of what follows in this paper could also be adapted, for continuous-time systems. When a structured, stable, stabilizing controller (which makes this particular Youla parametrization possible) is unknown, a more general parametrization exists (Sabau and Martins (2012)) leading to a more general affine constraint on the parameter; adapting our results for this problem could be a topic of further research.

While the problem is convex, the domain is infinite-dimensional, and solving it is certainly not straightforward. This equivalence holds for arbitrary closed-loop norm in the objective, and when the norm of interest is instead the $\mathcal{H}_2$ -norm, several methods have been put forth for addressing the resulting problem, as discussed in Section 1.

We focus on how to approach the optimal solution when the objective of interest is instead the $\mathcal{H}_\infty$ -norm.

4.1 QI - Sparsity Constraints

For the case of sparsity constraints, it was shown in (Rotkowitz and Lall (2006b)) that a necessary and sufficient condition for quadratic invariance is

$$K_{ki}^{\text{bin}} G_{ij}^{\text{bin}} K_{jl}^{\text{bin}} (1 - K_{kl}^{\text{bin}}) = 0, \quad (3)$$

for all $i, l \in \{1, \dots, n_y\}$, and all $j, k \in \{1, \dots, n_u\}$.

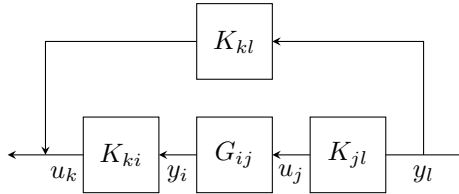


Fig. 3. QI interpretation for sparse controllers

An interpretation (see Figure 3) is that if a sensor measurement (y_l) can indirectly effect a control input (u_k) through the plant, then that controller must be able to directly observe that measurement ($K_{kl}^{\text{bin}} = 1$). This is closely related to the notion of partial nestedness (Ho and Chu (1972); Rotkowitz (2008)), and many problems of interest either fall in this class or can be relaxed or approximated to fall in this class.

5. Q-PARAMETRIZATION

In this section, we discuss a method for addressing the convex infinite-dimensional model-matching problem (2), known as Q -parametrization. This has long been used for the centralized problem (without the constraint) for objectives where more elegant solutions are not or were not available (including multiple-objective problems, cf. Hindi et al. (1998)) and has been suggested as a possible method for (2) since the QI results were first available. The idea is to use a finite-dimensional basis to parametrize the domain $\mathcal{RH}_\infty$, where the limit of the span will be dense in the original domain. In discrete-time, the usual choice of basis has a nice interpretation in the time-domain, as each basis vector corresponds to a finite impulse response of

a different delay and in a different part of the controller. Suppose we choose a maximum order of N for the map between each input and output of the controller parameter. Then for each $i \in \{1, \dots, n_u\}$ and $j \in \{1, \dots, n_y\}$, we have the approximate parametrization:

$$\hat{Q}_{ij}(z) = \sum_{k=0}^N \frac{\alpha_{ijk}}{z^k} \quad (4)$$

and there are $n_u \cdot n_y \cdot (N + 1)$ variables to find.

We can then state the following parametrized approximation to our convex decentralized model-matching problem (2):

$$\begin{aligned} &\text{minimize} \quad \|T_1 - T_2 \hat{Q} T_3\|_{\mathcal{H}_\infty} \\ &\text{subject to} \quad \hat{Q}_{ij}(z) = \sum_{k=0}^N \frac{\alpha_{ijk}}{z^k} \quad \forall i, j \\ &\quad \hat{Q} \in \mathcal{S} \end{aligned} \quad (5)$$

with variables $\hat{Q} \in \mathcal{RH}_\infty^{n_u \times n_y}$, $\alpha \in \mathbb{R}^{n_u \cdot n_y \cdot (N+1)}$, and assuming that we substitute the first constraint into the objective, we have a finite-dimensional convex optimization problem in the vector α .

We can find a state-space representation of $\hat{Q}$ as follows. For each $j \in \{1, \dots, n_y\}$, let $A_j^Q \in \mathbb{R}^{N \times N}$, and $B_j^Q \in \mathbb{R}^N$:

$$A_j^Q = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad B_j^Q = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and for each $i \in \{1, \dots, n_u\}$ and $j \in \{1, \dots, n_y\}$, let

$$C_{ij}^Q = [\alpha_{ijN} \cdots \alpha_{ij1}], \quad D_{ij}^Q = [\alpha_{ij0}].$$

Then define

$$\begin{aligned} A_Q &= \text{diag}(A_1^Q, \dots, A_{n_y}^Q), & B_Q &= \text{diag}(B_1^Q, \dots, B_{n_y}^Q), \\ C_Q &= \begin{bmatrix} C_{11}^Q & \cdots & C_{1n_y}^Q \\ \vdots & \ddots & \vdots \\ C_{n_u1}^Q & \cdots & C_{n_un_y}^Q \end{bmatrix} & D_Q &= \begin{bmatrix} D_{11}^Q & \cdots & D_{1n_y}^Q \\ \vdots & \ddots & \vdots \\ D_{n_u1}^Q & \cdots & D_{n_un_y}^Q \end{bmatrix} \end{aligned}$$

and we have

$$\hat{Q}(z) = \begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix} (z) \quad (6)$$

Remark 3. With this representation, all of the parameters α_{ijk} have been gathered in only C_Q and D_Q . This will allow the problem to be cast as one of finding an optimal static controller.

Remark 4. This representation shows that we can have different A_j^Q for different j , as long as the controller Q remains stable, and thus it is possible to have a different basis for each input.

Remark 5. With this representation, we get $\hat{Q}_{ij} = C_{ij}^Q(zI - A_j^Q)^{-1}B_j^Q + D_{ij}^Q$.

With the parameters all gathered in C_Q, D_Q , we now state a lemma showing how to impose the information constraint on these variables.

Lemma 6. If $\hat{Q} = \begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix}$, with A_Q, B_Q, C_Q, D_Q given as above, then $\hat{Q} \in \mathcal{S}$ if and only if

$$\begin{aligned} C_{ij}^Q &= 0 \text{ for all } (i, j) \text{ s.t. } K_{ij}^{\text{bin}} = 0 \\ D_{ij}^Q &= 0 \text{ for all } (i, j) \text{ s.t. } K_{ij}^{\text{bin}} = 0 \end{aligned} \quad (7)$$

Proof. It is straightforward to verify that the state-space representation gives $\hat{Q}_{ij}(z) = \sum_{k=0}^N \frac{\alpha_{ijk}}{z^k}$. It then follows that $\hat{Q} \in \mathcal{S}$ if and only if for all (i, j) such that $K_{ij}^{\text{bin}} = 0$, and for all $k \in \{0, 1, \dots, N\}$ we have that $\alpha_{ijk} = 0$, and this occurs if and only if (7) holds. ■

Remark 7. We note here how we would incorporate delay constraints. We would need to hold α_{ijk} to 0 for k below the number corresponding to the specified delay for $u_i \leftarrow y_j$, and would thus hold D_{ij}^Q and the necessary parts of C_{ij}^Q to be 0.

We can then form an equivalent optimization problem using this lemma and the approximated version of minimization problem (2) by replacing Q with $\hat{Q}$ and optimize over $C_Q \in \mathbb{R}^{n_u \times n_y N}$, and $D_Q \in \mathbb{R}^{n_u \times n_y}$, thus leaving the following finite-dimensional convex optimization problem

$$\begin{aligned} &\text{minimize} \quad \|T_1 - T_2 \hat{Q} T_3\|_{\mathcal{H}_\infty} \\ &\text{subject to} \quad \hat{Q} = \begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix} \\ &\quad C_{ij}^Q = 0 \text{ for all } (i, j) \text{ s.t. } K_{ij}^{\text{bin}} = 0 \\ &\quad D_{ij}^Q = 0 \text{ for all } (i, j) \text{ s.t. } K_{ij}^{\text{bin}} = 0 \end{aligned} \quad (8)$$

with variables $\hat{Q} \in \mathcal{RH}_\infty^{n_u \times n_y}$, $C_Q \in \mathbb{R}^{n_u \times n_y N}$, $D_Q \in \mathbb{R}^{n_u \times n_y}$, and assuming that we substitute the first constraint into the objective, we have a convex finite-dimensional problem in the matrices C_Q, D_Q .

5.1 Subgradient

We first address problem (5) directly, as, for a given value of variable α , we can compute the objective and can compute a subgradient. We can thus solve the parametrized optimal decentralized control problem with various methods. We demonstrate it here using the ellipsoid method. If evidence arises that this is remotely competitive with the performance of our main result, then more sophisticated algorithms will be explored for this direct approach.

The main ideas are from (Boyd and Barratt (1991)) and are adapted for the decentralized case. The advantage of the basic approach in this subsection, is that it can easily be adapted for other objectives of interest, such as other norms or multiple objectives.

Given a functional $f : \mathcal{X} \mapsto \mathbb{R}$, a subgradient of f at x_0 evaluated on y , denoted by $f^{\text{sg}}(x_0, y) : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$, is a linear functional in its second variable, such that:

$$f(y) \geq f(x) + f^{\text{sg}}(x_0, y) - f^{\text{sg}}(x_0, x_0), \quad \forall y \in \mathcal{X} \quad (9)$$

We want to obtain a subgradient of $\mathcal{H}_\infty$ -norm of closed loop map $T_1 - T_2 Q T_3$ when controller Q is approximated as $\hat{Q}(z) = \sum_{i,j} \sum_{k=0}^N \frac{\alpha_{ijk}}{z^k} \mathbf{E}_{ij}$, where $\mathbf{E}_{ij}$ is the zero matrix

with same size as Q except for its $(i, j)^{\text{th}}$ element which is one. By substituting $\hat{Q}$ for Q , close loop map can be written as a function of parameters $\alpha = \{\alpha_{ijk}\}$ as

$$\begin{aligned} H(\cdot) : \mathbb{R}^{n_u n_y (N+1)} &\mapsto \mathcal{RH}_\infty^{n_u \times n_y} \\ H(\alpha) &= T_1 - T_2 \left(\sum_{i,j} \sum_{k=0}^N \frac{\alpha_{ijk}}{z^k} \mathbf{E}_{ij} \right) T_3 \end{aligned} \quad (10)$$

We will derive the subgradient vector in two steps, first by describing a subgradient of $f(\alpha) = \|H(\alpha)\|_\infty$ at α^0 evaluated on α^1 , i.e. $f^{\text{sg}}(\alpha^0, \alpha^1)$, and then will derive the subgradient vector explicitly. Following will achieve the first step

Theorem 8. For $f(\alpha) = \|H(\alpha)\|_\infty$, a subgradient of f at α^0 evaluated on α^1 is given by

$$\begin{aligned} f^{\text{sg}}(\alpha^0, \alpha^1) &= \\ &- \Re \left(u_0^* T_2^{\omega_0} \left(\sum_{i,j} \sum_{k=0}^N \alpha_{ijk}^1 e^{-jk\omega_0} \mathbf{E}_{ij} \right) T_3^{\omega_0} v_0 \right) \end{aligned} \quad (11)$$

The equation is illustrated as follows, first we compute frequency ω_0 at which $\mathcal{H}_\infty$ norm of $H(\alpha^0)$ is achieved (i.e. $\sigma_{\max}(H(\alpha^0)(e^{j\omega_0})) = \|H(\alpha^0)(e^{j\omega_0})\|_{\mathcal{H}_\infty}$), then singular value decomposition of $H(\alpha^0)(e^{j\omega_0})$ would be computed to have $H(\alpha^0)(e^{j\omega_0}) = U S V^*$, then first columns of U and V are extracted and name as u_0 , and v_0 . Now we can form (11), where $T_i^{\omega_0} = T_i(e^{j\omega_0})$, $i = 1, 2, 3$.

Proof. To prove that (11) is a subgradient of f , we should show that it satisfies the subgradient inequality (9), but first observe that based on definition we have

$$\begin{aligned} f(\alpha^0) &= \|H(\alpha^0)\|_\infty = \Re(u_0^* (H(\alpha^0)(e^{j\omega_0})) v_0) \\ &= \Re(u_0^* T_1^{\omega_0} v_0) + f^{\text{sg}}(\alpha^0, \alpha^0) \end{aligned} \quad (12)$$

Next we can see that $\forall \alpha^1 \in \mathbb{R}^{n_u n_y (N+1)}$:

$$\begin{aligned} f(\alpha^1) &= \text{ess} \sup_{\omega, \|u\|=\|v\|=1} \Re(u^* (H(\alpha^1)(e^{j\omega})) v) \\ &\geq \Re(u_0^* (H(\alpha^1)(e^{j\omega_0})) v_0) \\ &= \Re(u_0^* T_1^{\omega_0} v_0) + f^{\text{sg}}(\alpha^0, \alpha^1) \\ &\stackrel{(12)}{=} f(\alpha^0) - f^{\text{sg}}(\alpha^0, \alpha^0) + f^{\text{sg}}(\alpha^0, \alpha^1) \\ &\Rightarrow f(\alpha^1) \geq f(\alpha^0) + f^{\text{sg}}(\alpha^0, \alpha^1) - f^{\text{sg}}(\alpha^0, \alpha^0) \quad \forall \alpha^1 \end{aligned} \quad (13)$$

Now we can proceed by computing the subgradient vector, more specifically. We are looking for $\phi \in \mathbb{R}^{n_u n_y (N+1)}$ such that $\langle \phi, \alpha^1 \rangle = f^{\text{sg}}(\alpha^0, \alpha^1)$

Theorem 9. Subgradient vector ϕ is given by its elements as

$$\phi_{ijk} = -\Re([u_0^* T_2^{\omega_0}]_i [T_3^{\omega_0} v_0]_j e^{-jk\omega_0}) \quad (14)$$

where $[\cdot]_i$ denotes i^{th} element of a vector.

Proof. RHS of (11) is can be written as

$$\begin{aligned} (11) &= \langle \phi, \alpha^1 \rangle \\ &= - \sum_{i,j} \sum_{k=0}^N \alpha_{ijk}^1 \Re(u_0^* T_2^{\omega_0} \mathbf{E}_{ij} T_3^{\omega_0} v_0 e^{-jk\omega_0}) \\ &= - \sum_{i,j,k} \alpha_{ijk}^1 \Re([u_0^* T_2^{\omega_0}]_i [T_3^{\omega_0} v_0]_j e^{-jk\omega_0}) \\ &\Rightarrow \phi_{ijk} = -\Re([u_0^* T_2^{\omega_0}]_i [T_3^{\omega_0} v_0]_j e^{-jk\omega_0}) \end{aligned} \quad (15)$$

This subgradient vector ϕ is used in implementations of ellipsoid method in later sections.

6. LMI FOR $\mathcal{H}_\infty$ -NORM WITH Q-PARAMETRIZATION

This section contains our main result. We first review a key result in Section 6.1 establishing that finding the $\mathcal{H}_\infty$ -optimal static controller for certain plants, including those resulting from a Youla parametrization followed by a Q-parametrization, can be cast as a semi-definite program. We then show in Section 6.2 how this result can be used together with the quadratic invariance results and those in the prequel to cast the $\mathcal{H}_\infty$ -optimal decentralized control problem as an SDP.

6.1 Static Output Feedback

In this subsection, we review the main result of (Scherer (2000)), which will be crucial to achieve our main result.

Theorem 10. Consider generalized discrete-time plants with state-space realization that can be partitioned as follows:

$$\begin{bmatrix} \hat{A}_1 & \hat{A} & \hat{B}_{11} & \hat{B} \\ 0 & \hat{A}_2 & \hat{B}_{21} & 0 \\ \hat{C}_{11} & \hat{C}_{12} & \hat{D}_{11} & \hat{D}_{12} \\ 0 & \hat{C} & \hat{D}_{21} & 0 \end{bmatrix} \quad (16)$$

The optimal static output feedback controller K^{static} along with optimal $\mathcal{H}_\infty$ norm, can be found by solving the following SDP

$$\begin{aligned} & \text{minimize } \gamma \\ & \text{subject to } \begin{bmatrix} \tilde{X} & 0 & \tilde{A}^T & \tilde{C}^T \\ 0 & \gamma I & \tilde{B}^T & \tilde{D}^T \\ \tilde{A} & \tilde{B} & \tilde{X} & 0 \\ \tilde{C} & \tilde{D} & 0 & \gamma I \end{bmatrix} \succ 0 \end{aligned} \quad (17)$$

with variables $\gamma \in \mathbb{R}$, and real matrices of appropriate dimension K^{static} , $E = E^T$, S , and $R = R^T$, as well as $\tilde{A}$, $\tilde{X}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ which are given by the additional constraints

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} \hat{A}_1 E & \hat{A}_1 S + \hat{A} + \hat{B} K^{\text{static}} \hat{C} - S \hat{A}_2 \\ 0 & R \hat{A}_2 \end{bmatrix}, \\ \tilde{B} &= \begin{bmatrix} \hat{B}_{11} + \hat{B} K^{\text{static}} \hat{D}_{21} - S \hat{B}_{21} \\ R \hat{B}_{21} \end{bmatrix}, \\ \tilde{C} &= [\hat{C}_{11} E \quad \hat{C}_{12} - \hat{D}_{12} K^{\text{static}} \hat{C} + \hat{C}_{11} S], \\ \tilde{D} &= \hat{D}_{11} - \hat{D}_{12} K^{\text{static}} \hat{D}_{21}, \\ \tilde{X} &= \text{diag}(E, R), \end{aligned} \quad (18)$$

which all are affine in all of the variables.

Proof. See (Scherer (2000)). ■

The paper also notes that plants without a 22-block (where the controller inputs to the plant do not effect the measurements which the controller may act on) can be partitioned as in (16), and are thus amenable to optimal static feedback with this SDP.

6.2 LMI Formulation with Quadratic Invariance

In this subsection we state our main result, showing how the problem of finding the $\mathcal{H}_\infty$ -optimal decentralized controller for a QI problem, after using Q-parametrization, can be formulated as an SDP.

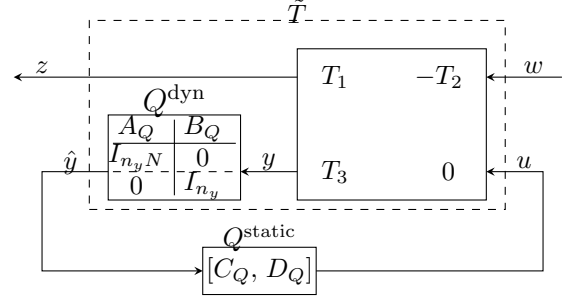


Fig. 4. $\tilde{T}$ defined by augmenting plant with fixed part of parametrized controller

The model-matching problem has (by definition) no 22-block, and can thus be represented as in (16). The parametrized controller that we are trying to design for it, was shown to be separable into a fixed dynamic part, which can be represented as

$$Q^{\text{dyn}} = \begin{bmatrix} A_Q & B_Q \\ I_{n_y N} & 0 \\ 0 & I_{n_y} \end{bmatrix},$$

and a variable static part, which can be given as the matrix $Q^{\text{static}} = [C_Q \ D_Q]$.

The fixed dynamic part (Q^{dyn}) can then be considered part of an augmented plant, as illustrated in Figure 4. This leaves us to optimize over static controllers (matrices) Q^{static} for the augmented plant $\tilde{T}$.

A state-space realization for $\tilde{T}$ is given by:

$$\begin{bmatrix} A_1 & 0 & 0 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & 0 & 0 & B_2 \\ 0 & 0 & A_3 & 0 & \tilde{B}_3 & 0 \\ 0 & 0 & B_Q C_3 & A_Q & B_Q D_3 & 0 \\ \tilde{C}_1 & -\tilde{C}_2 & 0 & 0 & \tilde{D}_1 & -\tilde{D}_2 \\ 0 & 0 & 0 & I_{n_y N} & 0 & 0 \\ 0 & 0 & C_3 & 0 & D_3 & 0 \end{bmatrix} \quad (19)$$

Where (A_i, B_i, C_i, D_i) is a state-space realization of T_i for $i = 1, 2, 3$. This partition still comports with (16). We can then apply the results of the previous subsection, and combine this with Lemma 6, to arrive at our main result.

Main result. The parametrized version of main problem (8) is solvable by the following SDP:

$$\begin{aligned} & \text{minimize } \gamma \\ & \text{subject to } \begin{bmatrix} \tilde{X} & 0 & \tilde{A}^T & \tilde{C}^T \\ 0 & \gamma I & \tilde{B}^T & \tilde{D}^T \\ \tilde{A} & \tilde{B} & \tilde{X} & 0 \\ \tilde{C} & \tilde{D} & 0 & \gamma I \end{bmatrix} \succ 0 \end{aligned} \quad (20)$$

$$C_{ij}^Q = 0 \quad \text{for } (i, j) \text{ s.t. } K_{ij}^{\text{bin}} = 0$$

$$D_{ij}^Q = 0 \quad \text{for } (i, j) \text{ s.t. } K_{ij}^{\text{bin}} = 0$$

with variables $\gamma \in \mathbb{R}$, and real matrices of appropriate dimension $Q^{\text{static}} = [C_Q \ D_Q]$, $E = E^T$, S , and $R =$

R^T , as well as $\tilde{A}$, $\tilde{X}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ which are given by the additional affine constraints (18), where in each constraint, the hat constants defined by (16), are set to their counterparts in (19), and the variable K^{static} is replaced by Q^{static} .

After obtaining the optimal $Q^{\text{static}} = [C_Q \ D_Q]$, we can recover the optimal $\hat{Q}$ as in (6). Then if we let $Q \approx \hat{Q}$, we can recover the approximated controller K by Theorem 17 in (Rotkowitz and Lall (2006b)).

7. NUMERICAL EXAMPLE

In this section, we consider some numerical examples. We use a discretized version of the same plant which was used in (Rotkowitz and Lall (2006b)) to demonstrate finding the $\mathcal{H}_2$ -optimal decentralized controllers, along with the same sequence of information constraints.

Consider an unstable lower triangular plant

$$G(z) = \begin{bmatrix} s(z) & 0 & 0 & 0 & 0 \\ s(z) & u(z) & 0 & 0 & 0 \\ s(z) & u(z) & s(z) & 0 & 0 \\ s(z) & u(z) & s(z) & s(z) & 0 \\ s(z) & u(z) & s(z) & s(z) & u(z) \end{bmatrix}$$

with $s(z) = \frac{0.1}{z-0.5}$, $u(z) = \frac{1}{z-2}$, and P given by

$$P_{11} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \quad P_{12} = \begin{bmatrix} G \\ I \end{bmatrix} \quad P_{21} = [G \ I]$$

and a sequence of sparsity constraints $K_1^{\text{bin}}, \dots, K_6^{\text{bin}}$

$$K_1^{\text{bin}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad K_2^{\text{bin}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$K_3^{\text{bin}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad K_4^{\text{bin}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$K_5^{\text{bin}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad K_6^{\text{bin}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

defining a sequence of information constraints $\mathcal{S}_i = \text{Sparse}(K_i^{\text{bin}})$ such that each subsequent constraint is less restrictive, and such that each is quadratically invariant under G . We also use $\mathcal{S}_7$ as the set of controllers with no sparsity constraints; i.e., the centralized case.

First, we apply our main result to the centralized problem, where we can compute the optimal solution with existing software. This serves as a sanity check to ensure that we get convergence to the optimum, and to explore how the parametrized solutions converge as the order grows.

Figure 5 plots the optimal $\mathcal{H}_\infty$ -norm obtained by solving our SDP (20) with no sparsity pattern, as the order of approximation N , increases from 1 to 13, which is solved by using `cvx` toolbox (CVX Research, Inc. (2012)) for MATLAB. It shows that as expected, as N increases from

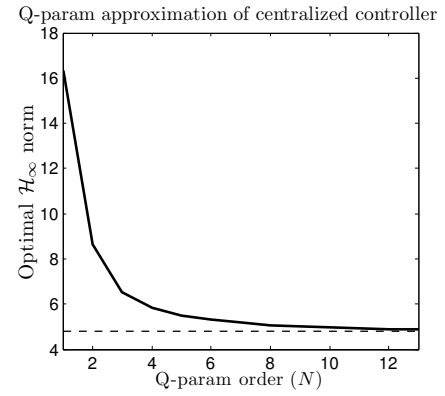


Fig. 5. Optimal norm versus order of estimation N for centralized controller

1 to 13, the optimal $\mathcal{H}_\infty$ -norm decreases and converges toward the actual solution, indicated by the dashed line, which was obtained using MATLAB's internal function `hinfscn`. The plant, and thus the actual optimal controller, are of order 5, and we see that we get convergence after increasing the order slightly beyond that.

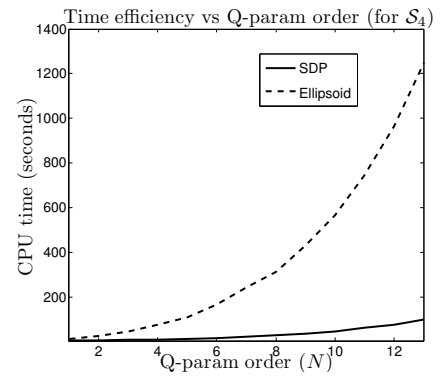


Fig. 6. Time efficiency of different methods for decentralized case $\mathcal{S}_4$

We then apply our results to the decentralized problems, Figure 6 shows how the two methods compare for the information constraint $\mathcal{S}_4$, as the order N again varies from 1 to 13. The SDP (20) is solved first, and the time it takes is shown with the solid line. The ellipsoid method is then used, with its stopping criterion chosen based on the optimal value of the SDP f_{SDP}^* , setting $\frac{f_{\text{ellip}}^{\text{best}} - f_{\text{SDP}}^*}{f_{\text{SDP}}^*} < 0.1$, such that we stop when we have an optimal point that is at least within 10% of the SDP solution. CPU time is reported from a machine with 2.3GHz CPU, and 8GB of RAM. Although SDP did always better than ellipsoid method for run of same algorithm on other random stable lower triangular plants, but their difference was not always in such a big scale.

We then turn our attention to computing and comparing the $\mathcal{H}_\infty$ -optimal solutions for the sequence of sparsity constraints. The results are presented in Figure 7 and are computed by solving the SDP (20) for K_i^{bin} , $i = 1, \dots, 6$, and $i = 7$ for centralized controller. In each case, N is fixed at 13. This shows, as expected, that as we relax the information constraint, the optimal norm would also be non-increasing, since $\mathcal{S}_i \subset \mathcal{S}_j$, for $i < j$.

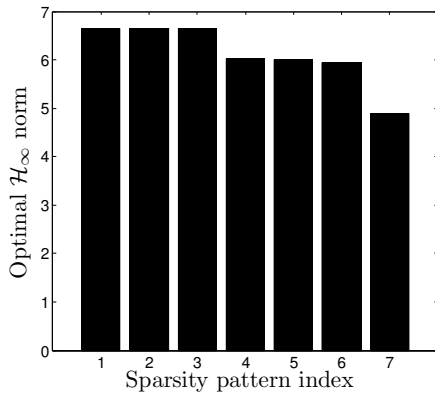


Fig. 7. Optimal norm for different sparsity patterns

Comparing these solutions to those for the $\mathcal{H}_2$ -norm in (Rotkowitz and Lall (2006b)), we see that there is again little-to-no difference from $\mathcal{S}_1$ to $\mathcal{S}_3$, as we allow the first measurement to be seen by the fourth and fifth controllers. There is again a jump in performance as we move to $\mathcal{S}_4$ and the fifth controller is able to see the third measurement. In the $\mathcal{H}_2$ case, there was little change to $\mathcal{S}_5$, but the norm decreased when moving to $\mathcal{S}_6$ as a lower-triangular controller is considered; for our $\mathcal{H}_\infty$ -norm there is very little gain from $\mathcal{S}_4$ to the lower triangular controller. In both cases, there is still a significant drop in the optimal norm from lower-triangular to fully centralized.

8. CONCLUSION

We considered the problem of finding $\mathcal{H}_\infty$ -optimal controllers subject to decentralization constraints, particularly for quadratically invariant constraints. We addressed the convex but infinite-dimensional problem that arises with Q -parametrization, allowing us to instead consider a sequence of finite-dimensional convex problems. We derived the subgradient for this problem, allowing many standard convex optimization techniques to be applied, and demonstrated this with an ellipsoid method. For our main result, we showed that this problem can be further converted into an SDP (20), with vastly superior performance.

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