Systems & Control Letters 58 (2009) 289–295

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Interconnections of nonlinear systems with "mixed" small gain and passivity properties and associated input–output stability results[☆]

Wynita M. Griggs[a,*,1], Brian D.O. Anderson[b], Alexander Lanzon[c,2], Michael C. Rotkowitz[d,2]

[a] Hamilton Institute, National University of Ireland, Maynooth, Co. Kildare, Ireland
[b] Department of Information Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia
[c] Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Sackville Street, Manchester M60 1QD, UK
[d] Department of Electrical and Electronic Engineering, Melbourne School of Engineering, The University of Melbourne, Melbourne VIC 3010, Australia

ARTICLE INFO

Article history:
Received 18 June 2007
Received in revised form
4 November 2008
Accepted 12 November 2008
Available online 19 December 2008

Keywords:
Nonlinear systems
Input–output stability
Finite gain
Passivity

ABSTRACT

A negative feedback interconnection consisting of two causal, nonlinear systems is shown to be input–output stable when a "mixed" small gain and passivity assumption is placed on each of the systems. The "mixed" small gain and passivity property captures the well-known notions of passivity and small gain associated with systems: the property can be appropriately reduced to an input and output strictly passive system description; or alternatively, can be reduced to a description of a system with small gain. More importantly, the property captures the concept of a "blending" of the small gain and passivity ideas. This concept of "blending" can be visualized, for example, by considering linear time-invariant systems that exhibit passivity-type properties at, say, low frequencies; and lose these passivity-type properties but have small gain at high frequencies.

1. Introduction

A desired property of a feedback interconnection of two nonlinear systems is that the interconnection is input–output stable [1]. To determine stability, one typically places assumptions on the two systems in the interconnection, and then shows that, if the closed-loop system's inputs belong to some class of functions, then the errors and outputs also belong to the same class of functions [2]. To illustrate, a negative feedback interconnection is shown in Fig. 1, where M_1 and M_2 are causal operators acting on the errors e_1 and e_2, respectively, to produce outputs y_1 and y_2, respectively.

The small gain and passivity theorems are two of the most important results in the input–output stability theory of interconnected systems. The small gain theorem states that if the product of the gains of two stable systems, interconnected as shown in Fig. 1, is less than one, then the feedback interconnection of the two systems is stable [2–5]. The passivity theorem guarantees stability of a feedback interconnection of two stable systems if, for instance, both of the systems are passive, and one of them is input strictly passive with finite gain [2–4,6]. However, there exist many situations where stability of an interconnection cannot be determined by use of the small gain or passivity theorems because the assumptions required on systems as stated in the theorems do not match the properties of the systems in the feedback interconnection in question.

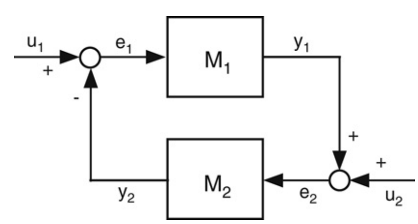

Fig. 1. Interconnection of M_1 and M_2.

In the papers [7,8], an input–output frequency domain stability result for the feedback interconnection of two stable, linear time-invariant (LTI) systems was provided. The assumption placed on the LTI systems was that they both exhibited a "mixed" small gain and passivity frequency domain property. That is, the small gain

☆ This work was supported by ARC Discovery-Projects Grants (DP0342683, DP0664427) and National ICT Australia. National ICT Australia is funded through the Australian Government's *Backing Australia's Ability* initiative, in part through the Australian Research Council.

* Corresponding author.
E-mail address: Wynita.Griggs@nuim.ie (W.M. Griggs).

1 W. Griggs contributed to this work during her Ph.D. candidature at The Australian National University and National ICT Australia.

2 A. Lanzon and M. Rotkowitz contributed to this work during their employment at The Australian National University and National ICT Australia.

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doi:10.1016/j.sysconle.2008.11.006

and passivity properties were "blended" in a way as to create a super-class of system assumptions (which captured, as subclasses, systems described by small gain concepts and systems described by passivity concepts). (Note also that systems not exhibiting the "mixed" small gain and passivity frequency domain property were dealt with in [7], via the introduction of weights to scale the interconnection such that the weighted systems then both did exhibit the property).

Obtaining results such as these has practical applicability. For instance, it has been observed that high frequency dynamics can frequently destroy the passivity property of an otherwise passive system. A celebrated controversy in adaptive control [9] depended on the observation that passivity conditions, normally forming part of the hypotheses used in the proofs of convergence of certain adaptive control algorithms, should not be assumed to be valid in practice (because high frequency dynamics often neglected for modelling purposes will always be present in a real system). Failure of the passivity condition invalidated the applicability of the associated theorem on the algorithm convergence to most real-life applications, and left a cloud hanging over the real-life use of the algorithm. Simulations of [9] confirmed that adverse behavior could occur when high frequency dynamics were explicitly taken into account.

The book [10] (see also [11,12]) described tools for establishing stability of adaptive systems of the type examined in [9]; that is, where "passivity" properties hold only for low frequency signals. Stability is established if additionally (and in a rough manner of speaking), "gains" are small at high frequencies (ie: a small gain property in the sense of the small gain theorem holds in the frequency band where the passivity condition fails). Thus, an important class of applications in which passivity and small gain ideas have to be "blended" exists.

In this paper, we aim to define a "mixed" small gain and passivity property, in the time domain, for causal nonlinear systems, and prove the associated input–output feedback interconnection stability result. It should be noted that integral quadratic constraints (IQCs) may provide an intermediate generalization between the LTI results reported in [7,8] and the nonlinear results that we seek in this paper. The stability theorem associated with IQCs described in [13, Theorem 1] captures the classical small gain and passivity/dissipativity theorems under the proviso that one of the two cascaded systems in the loop is linear and time-invariant. We divide the paper now into the following sections. Section 2 provides an intuitive link between the results obtained in [7,8] and the nonlinear results to be presented in this paper. In Section 3, the feedback interconnection under consideration is formally described. In Section 4, a "mixed" small gain and passivity property for a causal nonlinear system is defined. The feedback interconnection stability result is then discussed in Section 5. Conclusions are provided in Section 6.

2. Linear time-invariant results

Let us recall the notion of the "mixed" small gain and passivity frequency domain property as was described in [7,8]. Consider the negative feedback interconnection of two single-input, single-output (SISO) LTI systems with transfer functions $\hat{m}_1(s) = \frac{3}{(s+1)(s+2)}$ and $\hat{m}_2(s) = \frac{13}{(s+3)(s+4)}$. The Nyquist diagrams of these transfer functions are shown in Figs. 2 and 3, respectively. Since the product of the gains of the two systems is greater than one, the small gain theorem cannot be used as a tool to determine stability; and since the systems are not passive, stability cannot be guaranteed via the passivity theorem.

Notice that there exists a common frequency interval over which both systems are "passive" (and may or may not have "gain less than one"); and a common frequency interval over which both

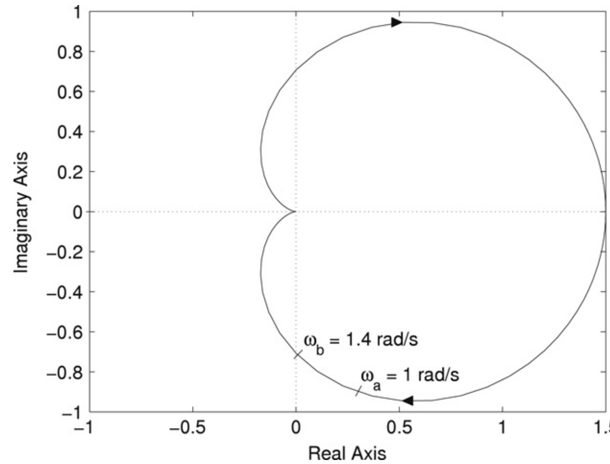

Fig. 2. Nyquist diagram of $m_1(s)$.

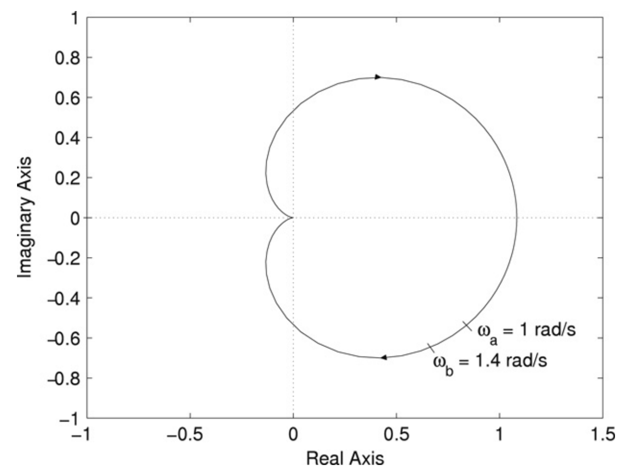

Fig. 3. Nyquist diagram of $m_1(s)$.

systems have "small gain" (and may or may not be "passive"). Systems exhibiting such "mixed" properties were mathematically described in [7,8] as follows: there exist constants $0 \leq \epsilon < 1$, $k > 0$ and $l > 0$ such that

$$-\langle \hat{m}\hat{f}, (k\alpha + 1 - \alpha)\hat{m}\hat{f}\rangle + 2\langle \hat{m}\hat{f}, \alpha\hat{f}\rangle - \langle \hat{f}, (l\alpha - \epsilon(1-\alpha))\hat{f}\rangle \geq 0 \quad (1)$$

$\forall \hat{f} \in \mathcal{H}_2$[3]; where $\hat{m} \in \mathcal{RH}_\infty$[4] and $\alpha(\omega)$ is a real, continuous, even function of frequency that is: (i) equal to one on frequency intervals for which the system described by $\hat{m}(s)$ is "input and output strictly passive"; (ii) equal to zero on frequency intervals for which the system described by $\hat{m}(s)$ has "gain less than one"; and (iii) is strictly greater than zero and strictly less than one on frequency intervals for which the system described by $\hat{m}(s)$ is "input and output strictly passive with gain less than one".[5] This description indeed captures the standard frequency domain concepts of passivity and small gain; it also captures a concept of "blending" of the passivity and small gain notions. In other words, the description indeed captures a class of systems that is larger

3 $\mathcal{H}_2$ is a subspace of $\mathcal{L}_2(j\mathbb{R})$ (the space of square integrable functions on $j\mathbb{R}$ including ∞) with functions analytic in $Re(s) > 0$, ie: in the open right-half plane.

4 $\mathcal{RH}_\infty$ is the space of proper, real-rational transfer function matrices of stable, continuous-time, LTI systems.

5 The notions of "input and output strict passivity" and "gain less than one" on a finite frequency interval were defined in [7,8].

than the class of passive systems together with the class of systems with small gain.

Shortly (in a subsequent section of this paper), a "mixed" small gain and passivity time domain property for nonlinear systems will be defined (which will lead us to the main business of the paper); immediately let us derive a time domain analog of the LTI "mixed" small gain and passivity frequency domain property. This will aid in motivating the definition of the "mixed" property for nonlinear systems. We first establish some preliminary mathematics and notation. The field of real numbers is denoted by $\mathbb{R}$. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are real inner product spaces. The inner product of $\mathcal{X}$ is denoted by $\langle\cdot,\cdot\rangle : \mathcal{X}\times\mathcal{X}\to\mathbb{R}$. A norm for each element of $\mathcal{X}$ is defined to be $\|f\|_{\mathcal{X}}^2 = \langle f, f\rangle$. An important property of inner product spaces is the Cauchy–Schwarz inequality; that is $|\langle f, g\rangle| \le \|f\|_{\mathcal{X}}\|g\|_{\mathcal{X}} \forall f, g \in \mathcal{X}$. Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. For a bounded linear operator $H : \mathcal{H}\to\mathcal{K}$, the Hilbert adjoint $H^{\sim} : \mathcal{K}\to\mathcal{H}$ of H is defined by $\langle Hh, k\rangle = \langle h, H^{\sim}k\rangle$ for all $h\in\mathcal{H}$ and $k\in\mathcal{K}$.

Let $\mathcal{L}_2[0,\infty)$ denote the Lebesgue space with inner product defined as

$$\langle f, g\rangle = \int_0^\infty g'(t)f(t)\mathrm{d}t,$$

where the superscript $(\cdot)'$ denotes the vector transpose. The norm of functions in $\mathcal{L}_2[0,\infty)$ is denoted by $\|\cdot\|$. For $T \in [0,\infty)$, let P_T denote the truncation operator. That is, for a function $f(t)$, $0 \le t < \infty$,

$$(P_T f)(t) := \begin{cases} f(t), & t \le T \\ 0, & t > T. \end{cases}$$

For convenience, the notation $f_T := P_T f$ will be used. We define $\langle f, g\rangle_T := \langle f_T, g_T\rangle$ and note that $\langle f_T, g_T\rangle = \langle f_T, g\rangle = \langle f, g_T\rangle$. Let $\mathcal{L}_{2e}$ denote the extension of the space $\mathcal{L}_2[0,\infty)$, defined by $\mathcal{L}_{2e} := \{f : f_T \in \mathcal{L}_2[0,\infty)\ \forall T \in [0,\infty)\}$. Recall that the space $\mathcal{L}_2[0,\infty)$ satisfies the following properties:

(i) The space $\mathcal{L}_2[0,\infty)$ is such that if $f \in \mathcal{L}_2[0,\infty)$, then $f_T \in \mathcal{L}_2[0,\infty)\ \forall T \in [0,\infty)$; and moreover, the space $\mathcal{L}_2[0,\infty)$ is such that $f = \lim_{T\to\infty} f_T$. Equivalently, the space $\mathcal{L}_2[0,\infty)$ is closed under the family of projections $\{P_T\}$.
(ii) If $f \in \mathcal{L}_2[0,\infty)$ and $T \in [0,\infty)$, then $\|f_T\| \le \|f\|$. Moreover, $\|f_T\|$ is a nondecreasing function of $T \in [0,\infty)$.
(iii) If $f \in \mathcal{L}_{2e}$, then $f \in \mathcal{L}_2[0,\infty)$ if and only if $\lim_{T\to\infty}\|f_T\| < \infty$.

The term *system* will be used to refer to a mapping from $\mathcal{L}_{2e}$ into $\mathcal{L}_{2e}$, which satisfies a causality condition. An operator $M : \mathcal{L}_{2e}\to\mathcal{L}_{2e}$ is causal if $P_T M P_T = P_T M$ for all $T \in [0,\infty)$. An operator $M : \mathcal{L}_{2e}\to\mathcal{L}_{2e}$ is anticausal if $(I - P_T)M(I - P_T) = (I - P_T)M$ for all $T \in [0,\infty)$. A system mapping $\mathcal{L}_{2e}$ into $\mathcal{L}_{2e}$ is input–output $\mathcal{L}_2$-stable if the output belongs to $\mathcal{L}_2[0,\infty)$ whenever the input belongs to $\mathcal{L}_2[0,\infty)$. For simplicity, input–output $\mathcal{L}_2$-stability will be referred to as input–output stability, or stability, when the context is clear. It is assumed that all systems considered are relaxed systems (that is, they have zero initial state). The operator $I : \mathcal{X}\to\mathcal{X}$, defined by $Ix := x$ for all $x\in\mathcal{X}$, denotes the identity operator. The operator $\mathbf{0} : \mathcal{X}\to\mathcal{Y}$, defined by $\mathbf{0}x := 0$ for all $x\in\mathcal{X}$ (where 0 denotes the zero vector from $\mathcal{Y}$), denotes the zero operator.

We now provide the following brief description of a (not necessarily finite-dimensional) LTI system (in the context of the input–output theory of systems) which may be found in texts such as [2,3,14,15] and in [16,17]. The discussion is limited to SISO systems for simplicity. Let $\mathcal{A}$ denote the set of generalized functions of the form

$$m(t) = \begin{cases} m_0\delta(t) + m_a(t), & t \ge 0 \\ 0, & t < 0 \end{cases}$$

where $m_0 \in \mathbb{R}$, $\delta(\cdot)$ denotes the unit impulse, and $m_a(\cdot)$ is such that

$$\int_0^\infty |m_a(\tau)|\mathrm{d}\tau < \infty.$$

Let $\hat{\mathcal{A}}$ denote the set consisting of all functions that are Laplace transforms of elements of $\mathcal{A}$. An LTI system $M : \mathcal{L}_2[0,\infty)\to\mathcal{L}_2[0,\infty)$ is defined to be a convolution operator of the form

$$(Mf)(t) = m(t) * f(t) = \int_{-\infty}^{\infty} m(\tau)f(t-\tau)\mathrm{d}\tau$$
$$= \int_{-\infty}^{\infty} m(t-\tau)f(\tau)\mathrm{d}\tau \quad (2)$$

where $m(\cdot) \in \mathcal{A}$ [2, Section D.1]. The function $m(\cdot)$ is called the kernel, or the impulse response, of the operator M. Furthermore, since $m(\tau) = 0$ for $\tau < 0$ and $f(t) = 0$ for $t < 0$, from (2)

$$(Mf)(t) = m_0 f(t) + \int_0^t m_a(\tau)f(t-\tau)\mathrm{d}\tau$$
$$= m_0 f(t) + \int_0^t m_a(t-\tau)f(\tau)\mathrm{d}\tau.$$

Then $\hat{m}(j\omega)$ as in (1) is the Fourier transform of $m(t)$; and let $\hat{f}(j\omega)$ denote the Fourier transform of input signal $f(t)$.

Suppose we introduce causal, bounded, linear operators $\Gamma : \mathcal{L}_2[0,\infty)\to\mathcal{L}_2[0,\infty)$ and $\mathrm{B} : \mathcal{L}_2[0,\infty)\to\mathcal{L}_2[0,\infty)$, where

$$\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B} = I. \quad (3)$$

Furthermore, suppose that Γ and B are time-invariant operators. Let $\gamma(\cdot)$ and $\beta(\cdot)$ denote the kernels of Γ and B, respectively, such that $\gamma(\cdot), \beta(\cdot) \in \mathcal{A}$. If $h^a(t) := h(-t)$ denotes the kernel of an anticausal LTI system, then

$$\gamma^a(t) * \gamma(t) + \beta^a(t) * \beta(t) = \delta(t) \quad (4)$$

from (3). (Recall that, if H is a linear causal operator, then its adjoint $H^{\sim}$ is anticausal [18].) Let $\hat{\gamma}(j\omega)$ and $\hat{\beta}(j\omega)$ denote the Fourier transforms of $\gamma(t)$ and $\beta(t)$, respectively. Then $\hat{\gamma}(-j\omega)\hat{\gamma}(j\omega) + \hat{\beta}(-j\omega)\hat{\beta}(j\omega) = 1$, since the kernel of the adjoint of a linear (causal) system is obtained by replacing $j\omega$ by $-j\omega$ when the kernel is expressed in terms of its Fourier transform. For convenience, let $(\cdot)^*(j\omega) := (\cdot)(-j\omega)$.

Now that we have defined $\hat{\gamma}(j\omega)$ and $\hat{\beta}(j\omega)$, we return to (1) and set $\alpha(\omega) = \hat{\beta}^*(j\omega)\hat{\beta}(j\omega)$. Rewriting (1) gives

$$-\langle\hat{m}\hat{f}, (k\hat{\beta}^*\hat{\beta} + \hat{\gamma}^*\hat{\gamma})\hat{m}\hat{f}\rangle + 2\langle\hat{m}\hat{f}, \hat{\beta}^*\hat{\beta}\hat{f}\rangle$$
$$-\langle\hat{f}, (l\hat{\beta}^*\hat{\beta} - \epsilon\hat{\gamma}^*\hat{\gamma})\hat{f}\rangle \ge 0,$$

which is identical to

$$-\langle\hat{m}\hat{f}, \hat{\gamma}^*\hat{\gamma}\hat{m}\hat{f}\rangle + \epsilon\langle\hat{f}, \hat{\gamma}^*\hat{\gamma}\hat{f}\rangle - k\langle\hat{m}\hat{f}, \hat{\beta}^*\hat{\beta}\hat{m}\hat{f}\rangle + 2\langle\hat{m}\hat{f}, \hat{\beta}^*\hat{\beta}\hat{f}\rangle$$
$$-l\langle\hat{f}, \hat{\beta}^*\hat{\beta}\hat{f}\rangle \ge 0.$$

Via the Paley–Wiener theorem [15, Theorem A.6.21], we can write

$$-\langle Mf, \Gamma^{\sim}\Gamma Mf\rangle + \epsilon\langle f, \Gamma^{\sim}\Gamma f\rangle - k\langle Mf, \mathrm{B}^{\sim}\mathrm{B}Mf\rangle + 2\langle Mf, \mathrm{B}^{\sim}\mathrm{B}f\rangle$$
$$-l\langle f, \mathrm{B}^{\sim}\mathrm{B}f\rangle \ge 0,$$

which is identical to

$$-\langle\Gamma Mf, \Gamma Mf\rangle + \epsilon\langle\Gamma f, \Gamma f\rangle - k\langle\mathrm{B}Mf, \mathrm{B}Mf\rangle + 2\langle\mathrm{B}Mf, \mathrm{B}f\rangle$$
$$-l\langle\mathrm{B}f, \mathrm{B}f\rangle \ge 0. \quad (5)$$

Inequality (5) provides us with a time domain version of the LTI "mixed" small gain and passivity property: given the existence of causal, bounded, LTI operators $\Gamma : \mathcal{L}_2[0,\infty)\to\mathcal{L}_2[0,\infty)$ and $\mathrm{B} : \mathcal{L}_2[0,\infty)\to\mathcal{L}_2[0,\infty)$ such that (3) is satisfied, a causal LTI system $M : \mathcal{L}_2[0,\infty)\to\mathcal{L}_2[0,\infty)$ has a "mixed" small gain and passivity property associated with it if there exist constants $0 \le \epsilon < 1$, $k > 0$ and $l > 0$ such that (5) holds for all $f \in \mathcal{L}_2[0,\infty)$.

3. Feedback system description

We want to derive an input–output stability result concerning the feedback interconnection shown in Fig. 1. This feedback interconnection is described by the equations

$$e_1 = u_1 - y_2 \quad y_1 = M_1 e_1$$
$$e_2 = u_2 + y_1 \quad y_2 = M_2 e_2$$

where $u_1, u_2 \in \mathcal{L}_{2e}$ are the (external) input signals; $e_1, e_2 \in \mathcal{L}_{2e}$ are the error signals; and $y_1, y_2 \in \mathcal{L}_{2e}$ are the output signals. The operators M_1 and M_2 are assumed to causally map $\mathcal{L}_{2e}$ into $\mathcal{L}_{2e}$. Furthermore, M_1 and M_2 each have associated with them a "mixed" small gain and passivity property (defined formally in the next section).

Strictness and non-strictness of the "mixed" small gain and passivity property will be dealt with formally in later sections. Similarly to the passivity and small gain theorems, one of the systems in the feedback interconnection is required to have a strict form of the "mixed" small gain and passivity property associated with it.

Well-posedness of the feedback interconnection corresponds to the existence and uniqueness of solutions e_1, e_2 and y_1, y_2 for each choice of u_1, u_2; and furthermore, requires that e_1, e_2 and y_1, y_2 depend causally on u_1, u_2 [19]. It is usual to also require that e_1, e_2 and y_1, y_2 depend, on finite intervals, Lipschitz continuously on u_1, u_2 [14]. References [1,20] describe conditions to impose on the operators M_1 and M_2 to guarantee well-posedness of the feedback-loop. We do not discuss well-posedness further in this paper; well-posedness of the feedback interconnection under consideration is assumed.

4. The "mixed" small gain and passivity property

We seek to formally define what we refer to as the "mixed" small gain and passivity property associated with a system. As mentioned previously, the "mixed" small gain and passivity property can be thought of as a "blending" of the concepts of passivity and small gain. The concepts of finite gain and passivity are defined for nonlinear systems, in the time domain, below.

Definition 1 (*[3]*). A system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to have a finite gain if there exist constants $\bar{\epsilon} \geq 0$ and $\eta \geq 0$, such that

$$\|(Mf)_T\| \leq \bar{\epsilon}\|f_T\| + \eta \tag{6}$$

for all input signals $f \in \mathcal{L}_{2e}$ and all $T \in [0, \infty)$.

The constant η is called the bias term and is included to allow for the case where $Mf \neq 0$ when $f = 0$ [3]. Clearly, if there do exist constants $\bar{\epsilon}$ and η such that (6) holds, then $\bar{\epsilon}$ is not uniquely defined. We call the gain of M the number ϵ defined by

$$\epsilon = \inf\{\bar{\epsilon} \in \mathbb{R}_+ : \exists \eta \text{ such that inequality (6) holds}\}$$

(see [2, Section III.2]). If $\epsilon < 1$, then the system M is said to have gain less than one; if $\epsilon \leq 1$, then M is said to have gain less than or equal to one. Systems with finite gain are said to be finite-gain stable [3]. Obviously, if a system has finite gain, then the system is input–output stable.

Definition 2 (*[3]*). A system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to be input and output strictly passive if there exist constants $k, l > 0$ such that

$$\langle Mf, f\rangle_T \geq k\|(Mf)_T\|^2 + l\|f_T\|^2 \tag{7}$$

for all input signals $f \in \mathcal{L}_{2e}$ and all $T \in [0, \infty)$, given that the system has zero initial state. The system M is said to be input strictly passive if it satisfies (7) with $k = 0$; output strictly passive if it satisfies (7) with $l = 0$; and, passive if it satisfies (7) with $k = l = 0$.

Note that input and output strict passivity is equivalent to input strict passivity with finite gain [6,21,22]. The (strict version of the) "mixed" small gain and passivity property is now defined below.

Definition 3. Let $\Gamma : \mathcal{L}_2[0, \infty) \to \mathcal{L}_2[0, \infty)$ and $\mathrm{B} : \mathcal{L}_2[0, \infty) \to \mathcal{L}_2[0, \infty)$ be any causal, bounded, linear (and not necessarily time-invariant) operators such that

$$\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B} = I. \tag{8}$$

Then a system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is said to have a strict "mixed" small gain and passivity property if there exist constants $0 \leq \epsilon < 1$, $k > 0$, $l > 0$ and $\eta \geq 0$ such that

$$\begin{aligned} &-\langle \Gamma(Mf)_T, \Gamma(Mf)_T\rangle + \epsilon\langle \Gamma f_T, \Gamma f_T\rangle - k\langle \mathrm{B}(Mf)_T, \mathrm{B}(Mf)_T\rangle \\ &\quad + 2\langle \mathrm{B}(Mf)_T, \mathrm{B}f_T\rangle - l\langle \mathrm{B}f_T, \mathrm{B}f_T\rangle + \eta \geq 0 \end{aligned} \tag{9}$$

for all input signals $f \in \mathcal{L}_{2e}$ and all $T \in [0, \infty)$.

The term η has been included to allow for output bias (that is, when zero system input does not imply zero system output) [23]. The "mixed" small gain and passivity property indeed captures the concepts of passivity or small gain normally associated with a system. If $\Gamma = \mathbf{0}$, then (9) describes an input and output strictly passive system. If $\mathrm{B} = \mathbf{0}$, then (9) describes a system with gain less than one. The description of the "mixed" small gain and passivity property additionally captures a concept of "blending" of the small gain and passivity ideas. In the case of LTI M, Γ and B for example, if Γ is time-invariant with $|\gamma(j\omega)|$ close to 0 at low frequencies and close to 1 at high frequencies, then the mixed property in qualitative terms corresponds to the system being passive at low frequencies and having small gain at high frequencies. (Recall that the papers [7,8] extensively illustrated the concept of "blending" of the small gain and passivity ideas in the frequency domain.)

The following observation can be made in regards to LTI systems. Suppose that the causal, bounded, linear operators Γ, $\mathrm{B} : \mathcal{L}_2[0, \infty) \to \mathcal{L}_2[0, \infty)$ are time-invariant. Consider a causal, LTI system $M : \mathcal{L}_2[0, \infty) \to \mathcal{L}_2[0, \infty)$. In this case, if M satisfies condition (9), then M also satisfies condition (5). To see this, first note that since M is linear (and has zero initial state), there is no loss in generality in setting $\eta = 0$ in (9) [24], giving

$$\begin{aligned} &-\langle \Gamma(Mf)_T, \Gamma(Mf)_T\rangle + \epsilon\langle \Gamma f_T, \Gamma f_T\rangle - k\langle \mathrm{B}(Mf)_T, \mathrm{B}(Mf)_T\rangle \\ &\quad + 2\langle \mathrm{B}(Mf)_T, \mathrm{B}f_T\rangle - l\langle \mathrm{B}f_T, \mathrm{B}f_T\rangle \geq 0. \end{aligned} \tag{10}$$

Now assuming that M satisfies (10), consider an arbitrary input $f \in \mathcal{L}_2[0, \infty)$ and note that, if $f \in \mathcal{L}_2[0, \infty)$, then $f_T \in \mathcal{L}_2[0, \infty)$ for all $T \in [0, \infty)$. Since $f \in \mathcal{L}_2[0, \infty)$ and $M, \Gamma, \mathrm{B} : \mathcal{L}_2[0, \infty) \to \mathcal{L}_2[0, \infty)$, we can take limits as $T \to \infty$ to obtain (5). Note that (10) is a stronger condition than (5). That is, since (10) implies (5) (in the case of stable LTI systems), (10) is a more demanding condition.

A consequence of a system having a strict "mixed" small gain and passivity property, as defined in Definition 3, is that the system is guaranteed to have finite gain.

Lemma 4. *A system $M : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ with a strict "mixed" small gain and passivity property (in the sense of Definition 3) has finite gain.*

Proof. Inequality (9) can be rewritten as

$$\begin{aligned} &\langle (Mf)_T, \Gamma^{\sim}\Gamma(Mf)_T\rangle + k\langle (Mf)_T, \mathrm{B}^{\sim}\mathrm{B}(Mf)_T\rangle \\ &\quad \leq \epsilon\langle f_T, \Gamma^{\sim}\Gamma f_T\rangle - l\langle f_T, \mathrm{B}^{\sim}\mathrm{B}f_T\rangle + 2\langle (Mf)_T, \mathrm{B}^{\sim}\mathrm{B}f_T\rangle + \eta \\ &\quad \leq \epsilon\langle f_T, \Gamma^{\sim}\Gamma f_T\rangle + l\langle f_T, \mathrm{B}^{\sim}\mathrm{B}f_T\rangle + 2\langle (Mf)_T, \mathrm{B}^{\sim}\mathrm{B}f_T\rangle + \eta. \end{aligned} \tag{11}$$

Let $\phi = \min\{1, k\}$, so that the first term of the above inequality is greater than or equal to

$$\begin{aligned} &\phi\left(\langle (Mf)_T, \Gamma^{\sim}\Gamma(Mf)_T\rangle + \langle (Mf)_T, \mathrm{B}^{\sim}\mathrm{B}(Mf)_T\rangle\right) \\ &\quad = \phi\langle (Mf)_T, (\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B})(Mf)_T\rangle \\ &\quad = \phi\langle (Mf)_T, (Mf)_T\rangle \end{aligned}$$

using (8). That is, the first term of inequality (11) is greater than or equal to $\phi\|(Mf)_T\|^2$.

Let $\psi = \max\{\epsilon, l\}$, so that the last term of inequality (11) is less than or equal to

$$\begin{aligned}
&\psi\left(\langle f_T, \Gamma^{\sim}\Gamma f_T\rangle + \langle f_T, \mathrm{B}^{\sim}\mathrm{B}f_T\rangle\right) + 2\langle (Mf)_T, \mathrm{B}^{\sim}\mathrm{B}f_T\rangle + \eta\\
&\quad = \psi\langle f_T, (\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B})f_T\rangle + 2\langle (Mf)_T, \mathrm{B}^{\sim}\mathrm{B}f_T\rangle + \eta\\
&\quad = \psi\|f_T\|^2 + 2\langle (Mf)_T, \mathrm{B}^{\sim}\mathrm{B}f_T\rangle + \eta \quad \text{(using (8))}\\
&\quad \le \psi\|f_T\|^2 + 2\|(Mf)_T\|\|\mathrm{B}^{\sim}\mathrm{B}\|\|f_T\| + \eta\\
&\quad\ \text{(using the Cauchy–Schwarz and submultiplicative inequalities)}\\
&\quad \le \psi\|f_T\|^2 + 2\|(Mf)_T\|\|f_T\| + \eta \quad (\text{since } \|\mathrm{B}^{\sim}\mathrm{B}\| \le 1).
\end{aligned}$$

Since $\phi > 0$ we can conclude that

$$\|(Mf)_T\|^2 \le 2\bar{\phi}\|f_T\|\|(Mf)_T\| + \bar{\phi}\left(\psi\|f_T\|^2 + \eta\right)$$

where $\bar{\phi} := \frac{1}{\phi}$; and so

$$\begin{aligned}
\|(Mf)_T\| &\le \bar{\phi}\|f_T\| + \sqrt{\bar{\phi}^2\|f_T\|^2 + \bar{\phi}\left(\psi\|f_T\|^2 + \eta\right)}\\
&\le \bar{\phi}\|f_T\| + \sqrt{\bar{\phi}^2\|f_T\|^2 + \bar{\phi}\psi\|f_T\|^2} + \sqrt{\bar{\phi}\eta}\\
&= \left(\bar{\phi} + \sqrt{\bar{\phi}^2 + \bar{\phi}\psi}\right)\|f_T\| + \sqrt{\bar{\phi}\eta}. \quad \Box
\end{aligned}$$

5. Stability result of feedback interconnection

An input–output stability result for the feedback interconnection shown in Fig. 1 is now provided. The result states that, if systems M_1 and M_2 each have associated with them a "mixed" small gain and passivity property, and furthermore, if the "mixed" small gain and passivity property associated with M_2 is strict, then the feedback interconnection is stable.[6]

Theorem 5. *Consider a feedback interconnection as shown in Fig. 1 and described by the equations*

$$e_1 = u_1 - M_2 e_2 \tag{12}$$
$$e_2 = u_2 + M_1 e_1 \tag{13}$$

where M_1 and M_2 causally map $\mathcal{L}_{2e}$ into $\mathcal{L}_{2e}$. Assume that for any u_1 and u_2 in $\mathcal{L}_2[0,\infty)$, there are solutions e_1 and e_2 in $\mathcal{L}_{2e}$. Suppose that there exist constants $\epsilon_1, k_1, l_1, \eta_1, \epsilon_2, k_2, l_2$ and η_2 such that

$$\begin{aligned}
&-\langle \Gamma(M_1f)_T, \Gamma(M_1f)_T\rangle + \epsilon_1\langle \Gamma f_T, \Gamma f_T\rangle - k_1\langle B(M_1f)_T,\\
&\quad B(M_1f)_T\rangle + 2\langle B(M_1f)_T, Bf_T\rangle - l_1\langle Bf_T, Bf_T\rangle + \eta_1 \ge 0
\end{aligned}\tag{14}$$
$$\begin{aligned}
&-\langle \Gamma(M_2f)_T, \Gamma(M_2f)_T\rangle + \epsilon_2\langle \Gamma f_T, \Gamma f_T\rangle - k_2\langle B(M_2f)_T,\\
&\quad B(M_2f)_T\rangle + 2\langle B(M_2f)_T, Bf_T\rangle - l_2\langle Bf_T, Bf_T\rangle + \eta_2 \ge 0
\end{aligned}\tag{15}$$

$\forall f \in \mathcal{L}_{2e}, \forall T \in [0,\infty)$, where Γ and B are as defined in Definition 3. Under these conditions, if

$$\begin{aligned}
&0 \le \epsilon_1 \le 1 \quad l_1 + k_2 \ge 0\\
&0 \le \epsilon_2 < 1 \quad l_2 + k_1 > 0\\
&k_2 > 0, \quad l_2 > 0
\end{aligned}$$

then $u_1, u_2 \in \mathcal{L}_2[0,\infty)$ imply that $e_1, e_2, M_1e_1, M_2e_2 \in \mathcal{L}_2[0,\infty)$.

To avoid confusion, note that Γ as it appears in (14) and (15) is the same operator. Similarly, B as it appears in (14) and (15) is the same operator.[7] Thus M_1 and M_2 satisfy the same "mixed" small gain and passivity condition as far as frequency dependency is concerned; the constants ϵ_i, k_i, l_i and η_i may differ for $i = 1, 2$. Also note that with appropriate choices of Γ and B, Theorem 5 reduces to the passivity theorem ($\Gamma = \mathbf{0}$) and the small gain theorem (B$= \mathbf{0}$), respectively.

The input and output signal space for the feedback interconnection shown in Fig. 1 is the product space $\mathcal{L}_{2e} \times \mathcal{L}_{2e}$, and the elements of the input and output signal space are $u := \binom{u_1}{u_2}$ and $y := \binom{y_1}{y_2}$, respectively. Inner products in these spaces are derived by summing inner products in the component spaces. We proceed with a proof for Theorem 5 by summing the inner products of (14) and (15) to derive an inner product inequality describing the feedback interconnection. (This is as opposed to having two separate inequalities, namely (14) and (15), describing the feedback interconnection's component systems, namely M_1 and M_2, respectively.) Then, appropriate manipulations of the inner product inequality give the desired stability result.

Proof. Truncating (12) and (13) gives

$$e_{1T} = u_{1T} - (M_2e_2)_T \tag{16}$$
$$e_{2T} = u_{2T} + (M_1e_1)_T. \tag{17}$$

For any $u_1, u_2 \in \mathcal{L}_2[0,\infty)$, for any $T \in [0,\infty)$,

$$\begin{aligned}
&-\langle \Gamma(M_1e_1)_T, \Gamma(M_1e_1)_T\rangle + \epsilon_1\langle \Gamma e_{1T}, \Gamma e_{1T}\rangle - k_1\langle \mathrm{B}(M_1e_1)_T,\\
&\quad \mathrm{B}(M_1e_1)_T\rangle + 2\langle \mathrm{B}(M_1e_1)_T, \mathrm{B}e_{1T}\rangle - l_1\langle \mathrm{B}e_{1T}, \mathrm{B}e_{1T}\rangle\\
&\quad + \eta_1 - \langle \Gamma(M_2e_2)_T, \Gamma(M_2e_2)_T\rangle + \epsilon_2\langle \Gamma e_{2T}, \Gamma e_{2T}\rangle\\
&\quad - k_2\langle \mathrm{B}(M_2e_2)_T, \mathrm{B}(M_2e_2)_T\rangle\\
&\quad + 2\langle \mathrm{B}(M_2e_2)_T, \mathrm{B}e_{2T}\rangle - l_2\langle \mathrm{B}e_{2T}, \mathrm{B}e_{2T}\rangle + \eta_2\\
&= -\langle \Gamma e_{2T} - \Gamma u_{2T}, \Gamma e_{2T} - \Gamma u_{2T}\rangle + \epsilon_1\langle \Gamma e_{1T}, \Gamma e_{1T}\rangle\\
&\quad - l_1\langle \mathrm{B}e_{1T}, \mathrm{B}e_{1T}\rangle + 2\langle \mathrm{B}e_{2T} - \mathrm{B}u_{2T}, \mathrm{B}e_{1T}\rangle\\
&\quad - k_1\langle \mathrm{B}e_{2T} - \mathrm{B}u_{2T}, \mathrm{B}e_{2T} - \mathrm{B}u_{2T}\rangle + \eta_1\\
&\quad - \langle \Gamma u_{1T} - \Gamma e_{1T}, \Gamma u_{1T} - \Gamma e_{1T}\rangle + \epsilon_2\langle \Gamma e_{2T}, \Gamma e_{2T}\rangle\\
&\quad - l_2\langle \mathrm{B}e_{2T}, \mathrm{B}e_{2T}\rangle + 2\langle \mathrm{B}u_{1T} - \mathrm{B}e_{1T}, \mathrm{B}e_{2T}\rangle\\
&\quad - k_2\langle \mathrm{B}u_{1T} - \mathrm{B}e_{1T}, \mathrm{B}u_{1T} - \mathrm{B}e_{1T}\rangle + \eta_2\\
&= -\langle e_{1T}, [(1-\epsilon_1)\Gamma^{\sim}\Gamma + (l_1 + k_2)\mathrm{B}^{\sim}\mathrm{B}]\, e_{1T}\rangle\\
&\quad - \langle u_{1T}, (\Gamma^{\sim}\Gamma + k_2\mathrm{B}^{\sim}\mathrm{B})u_{1T}\rangle - \langle e_{2T}, [(1-\epsilon_2)\Gamma^{\sim}\Gamma\\
&\quad + (l_2 + k_1)\mathrm{B}^{\sim}\mathrm{B}]e_{2T}\rangle - \langle u_{2T}, (\Gamma^{\sim}\Gamma + k_1\mathrm{B}^{\sim}\mathrm{B})u_{2T}\rangle\\
&\quad + 2\langle e_{1T}, (\Gamma^{\sim}\Gamma + k_2\mathrm{B}^{\sim}\mathrm{B})u_{1T}\rangle + 2\langle e_{2T}, (\Gamma^{\sim}\Gamma + k_1\mathrm{B}^{\sim}\mathrm{B})u_{2T}\rangle\\
&\quad - 2\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle + 2\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle + \eta_1 + \eta_2
\end{aligned}$$

using (16) and (17) to substitute for $(M_2e_2)_T$ and $(M_1e_1)_T$, respectively, and then rearranging. Using (14) and (15), the first, and thus the last, member of this equality is greater than or equal to zero. In this inequality, set $\bar{\eta} := \eta_1 + \eta_2$ for convenience. In other words, for any $u_1, u_2 \in \mathcal{L}_2[0,\infty)$ and any $T \in [0,\infty)$, we know that there exist constants $\epsilon_1, k_1, l_1, \epsilon_2, k_2, l_2$ and $\bar{\eta}$ such that

$$\begin{aligned}
&\langle e_{1T}, [(1-\epsilon_1)\Gamma^{\sim}\Gamma + (l_1 + k_2)\mathrm{B}^{\sim}\mathrm{B}]\, e_{1T}\rangle + \langle e_{2T}, [(1-\epsilon_2)\Gamma^{\sim}\Gamma\\
&\quad + (l_2 + k_1)\mathrm{B}^{\sim}\mathrm{B}]e_{2T}\rangle \le 2\langle e_{1T}, (\Gamma^{\sim}\Gamma + k_2\mathrm{B}^{\sim}\mathrm{B})u_{1T}\rangle\\
&\quad + 2\langle e_{2T}, (\Gamma^{\sim}\Gamma + k_1\mathrm{B}^{\sim}\mathrm{B})u_{2T}\rangle - 2\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle\\
&\quad + 2\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle - \langle u_{1T}, (\Gamma^{\sim}\Gamma + k_2\mathrm{B}^{\sim}\mathrm{B})u_{1T}\rangle\\
&\quad - \langle u_{2T}, (\Gamma^{\sim}\Gamma + k_1\mathrm{B}^{\sim}\mathrm{B})u_{2T}\rangle + \bar{\eta}
\end{aligned}\tag{18}$$

$\forall e_1, e_2 \in \mathcal{L}_{2e}, \forall T \in [0,\infty)$.

[6] In fact (corresponding to the choice of constants k_1 and l_1 below), the result permits M_1 to not have a "mixed" small gain and passivity property associated with it, provided that this "lack" of the property is compensated by the "strength" of the "mixed" small gain and passivity property associated with M_2. The constants defined in Theorem 5 and the conditions associated with them quantify these ideas of "lack", "strength" and compensation.

[7] In the LTI case, this relates to the requirement that a common frequency interval can be found on which both systems in the feedback interconnection are "input and output strictly passive and have gain less than one" (see [7,8] for details).

The left-hand side of inequality (18) is equal to

$$(1-\epsilon_1)\langle e_{1T}, \Gamma^{\sim}\Gamma e_{1T}\rangle + (l_1+k_2)\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}e_{1T}\rangle + (1-\epsilon_2)\langle e_{2T}, \Gamma^{\sim}\Gamma e_{2T}\rangle + (l_2+k_1)\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}e_{2T}\rangle,$$

which is greater than or equal to

$$(1-\epsilon_2)\langle e_{2T}, \Gamma^{\sim}\Gamma e_{2T}\rangle + (l_2+k_1)\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}e_{2T}\rangle \tag{19}$$

since $1-\epsilon_1, l_1+k_2 \geq 0$. Let $\sigma = \min\{1-\epsilon_2, l_2+k_1\}$ (noting that $\sigma > 0$) such that the term denoted by (19) is greater than or equal to

$$\sigma\left(\langle e_{2T}, \Gamma^{\sim}\Gamma e_{2T}\rangle + \langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}e_{2T}\rangle\right) = \sigma\langle e_{2T}, (\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B})e_{2T}\rangle = \sigma\|e_{2T}\|^2$$

using (8).

The right-hand side of inequality (18) is equal to

$$2\langle e_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle + 2k_2\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle + 2\langle e_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle + 2k_1\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle - 2\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle + 2\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle - \langle u_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle - k_2\langle u_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle - \langle u_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle - k_1\langle u_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle + \bar{\eta}$$

which is less than or equal to

$$2|\langle e_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle| + 2k_2|\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle| + 2|\langle e_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle| + 2|k_1||\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle| + 2|\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle| + 2|\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle| + \langle u_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle + k_2\langle u_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle + \langle u_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle + |k_1|\langle u_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle + \bar{\eta}. \tag{20}$$

Let $\rho = \max\{1, k_2, |k_1|\}$ such that the term denoted by (20) is less than or equal to

$$2\rho(|\langle e_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle| + |\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle| + |\langle e_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle| + |\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle|) + 2(|\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle| + |\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle|) + \rho(\langle u_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle + \langle u_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle + \langle u_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle + \langle u_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle) + \bar{\eta}. \tag{21}$$

Note that

$$\rho(\langle u_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle + \langle u_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle + \langle u_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle + \langle u_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle) = \rho(\langle u_{1T}, (\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B})u_{1T}\rangle + \langle u_{2T}, (\Gamma^{\sim}\Gamma + \mathrm{B}^{\sim}\mathrm{B})u_{2T}\rangle) = \rho(\|u_{1T}\|^2 + \|u_{2T}\|^2)$$

using (8). So the term denoted by (21) is equal to

$$2\rho(|\langle e_{1T}, \Gamma^{\sim}\Gamma u_{1T}\rangle| + |\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle| + |\langle e_{2T}, \Gamma^{\sim}\Gamma u_{2T}\rangle| + |\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle|) + 2(|\langle e_{1T}, \mathrm{B}^{\sim}\mathrm{B}u_{2T}\rangle| + |\langle e_{2T}, \mathrm{B}^{\sim}\mathrm{B}u_{1T}\rangle|) + \rho(\|u_{1T}\|^2 + \|u_{2T}\|^2) + \bar{\eta}$$

which is less than or equal to

$$2\rho(\|\Gamma^{\sim}\Gamma\| + \|\mathrm{B}^{\sim}\mathrm{B}\|)(\|e_{1T}\|\|u_{1T}\| + \|e_{2T}\|\|u_{2T}\|) + 2\|\mathrm{B}^{\sim}\mathrm{B}\|(\|e_{1T}\|\|u_{2T}\| + \|e_{2T}\|\|u_{1T}\|) + \rho(\|u_{1T}\|^2 + \|u_{2T}\|^2) + \bar{\eta} \tag{22}$$

by using the Cauchy–Schwarz and submultiplicative inequalities. Eliminating e_{1T} using (16), using the triangle inequality, and then applying Lemma 4 to M_2 shows that the term denoted by (22) is less than or equal to

$$2\rho(\|\Gamma^{\sim}\Gamma\| + \|\mathrm{B}^{\sim}\mathrm{B}\|)((\|u_{1T}\| + \kappa\|e_{2T}\| + \xi)\|u_{1T}\| + \|e_{2T}\|\|u_{2T}\|) + \bar{\eta} + 2\|\mathrm{B}^{\sim}\mathrm{B}\|((\|u_{1T}\| + \kappa\|e_{2T}\| + \xi)\|u_{2T}\| + \|e_{2T}\|\|u_{1T}\|) + \rho(\|u_{1T}\|^2 + \|u_{2T}\|^2)$$

where the non-negative constants κ and ξ exist due to the boundedness of M_2. Since $\sigma > 0$, we can conclude that

$$\|e_{2T}\|^2 \leq 2\bar{b}(T)\|e_{2T}\| + \bar{c}(T), \tag{23}$$

where $\bar{b}(T)$ and $\bar{c}(T)$ tend to finite values $\bar{b}$ and $\bar{c}$, respectively, as $T \to \infty$, since $u_1, u_2 \in \mathcal{L}_2[0,\infty)$. From (23)

$$\|e_{2T}\| \leq \bar{b}(T) + (\bar{b}(T)^2 + \bar{c}(T))^{\frac{1}{2}}$$

$\forall T \in [0,\infty)$, and remains bounded as $T \to \infty$. So $e_2 \in \mathcal{L}_2[0,\infty)$. From Lemma 4 the same holds for M_2e_2, ie: $M_2e_2 \in \mathcal{L}_2[0,\infty)$. By (12) and (13) it follows that $e_1, M_1e_1 \in \mathcal{L}_2[0,\infty)$. □

6. Conclusions

An input–output stability result was obtained for a standard feedback interconnection of two causal, nonlinear systems, where each system has a "mixed" small gain and passivity assumption associated with it. We indicated that the "mixed" small gain and passivity property reduces to a description of a system that is input and output strictly passive; or alternatively, to a description of a system that has gain less than one, when certain operators are appropriately defined. The "mixed" small gain and passivity property also captures a notion of "blending" of the small gain and passivity ideas, and thus describes a class of systems that is larger than the class of passive systems, together with the class of systems with small gain. In future work, we intend that relationships to Lyapunov stability and dissipativity concepts be explored. It will also be necessary to investigate techniques for determining whether or not a nonlinear system has a "mixed" small gain and passivity property associated with it.

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