Affine Controller Parameterization for Decentralized Control Over Banach Spaces

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***Abstract*—We cast the problem of optimal decentralized control as one of minimizing a closed-loop norm subject to a subspace constraint on the controller. In this note, we consider continuous linear operators on Banach spaces, and show that a simple property called quadratic invariance is necessary and sufficient for the constraint set to be preserved under feedback, and thus allows optimal synthesis to be recast as a convex optimization problem. These results hold for any norm and any Banach space.**

***Index Terms*—Banach spaces, bounded linear operators, convex optimization, decentralized control, networked control, robust control.**

I. Introduction

An important problem in control is that of constructing decentralized control systems, where instead of a single controller connected to a physical system, one has separate controllers, each with access to different information and with authority over different decision or actuation variables. Examples of such systems include formation flight of aircraft, the electricity distribution grid, and the control of smart structures. Standard controls analysis breaks down when one imposes decentralization on the controller.

There are many variations of this problem, depending on how the limited availability of information is specified, the structure of the physical systems, and whether and how separate controllers can communicate. In a standard controls framework, or a two-input two-output framework, the decentralization of the system manifests itself as structural constraints on the allowable controllers, such as sparsity or delay constraints. Therefore a canonical problem one would like to solve in decentralized control is to minimize a norm of the closed-loop map subject to a subspace constraint as follows:

$$\begin{array}{ll} \text{minimize} & \|f(P,K)\| \\ \text{subject to} & K \in S \end{array}.$$

For a general linear operator P and subspace S there is no known tractable algorithm for computing the optimal K. It has been known since 1968 [14] that even the simplest versions of this problem can be extremely difficult. In fact, certain cases have been shown to be computationally intractable [2], [7]. However, there are also several special cases of this problem for which efficient algorithms have been found [5], [8], [12], [13].

In this note, we show that if the constraints on the controller satisfy a simple condition, called *quadratic invariance*, with respect to the system being controlled, then the optimal decentralized control problem may be reduced to a convex optimization problem. This condition unifies previously identified tractable problems.

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Decentralized control has been addressed in many frameworks, with much important early work coming from team theory [5]. The formulation above provides a single framework for analyzing the decentralized control of systems static and dynamic, finite-dimensional and infinite-dimensional, stable and unstable, and elucidates why the same information structures yield tractable optimization problems regardless of the function spaces being considered. In [10] and [11], we considered causal linear time-invariant operators on extended linear spaces in this framework.

In this note, we consider the case of continuous linear operators on arbitrary Banach spaces, and show that quadratic invariance is necessary and sufficient for the constraint set to be preserved under feedback, provided that a technical condition is met. We then show that quadratic invariance thus allows for convex synthesis of optimal decentralized controllers. Finally, we provide an example which shows that, while the technical conditions are usually trivially satisfied, the result can fail if the technical conditions are not met.

A. Preliminaries

Given topological vector spaces $\mathcal{X}, \mathcal{Y}$, let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ such that T is linear and continuous. Note that if $\mathcal{X}, \mathcal{Y}$ are Banach spaces, then all such T are bounded. We abbreviate $\mathcal{L}(\mathcal{X}, \mathcal{X})$ with $\mathcal{L}(\mathcal{X})$.

Suppose $P \in \mathcal{L}(\mathcal{W} \times \mathcal{U}, \mathcal{Z} \times \mathcal{Y})$. Partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

so that $P_{11} : \mathcal{W} \to \mathcal{Z}$, $P_{12} : \mathcal{U} \to \mathcal{Z}$, $P_{21} : \mathcal{W} \to \mathcal{Y}$, and $P_{22} : \mathcal{U} \to \mathcal{Y}$. Suppose $K \in \mathcal{L}(\mathcal{Y}, \mathcal{U})$. If $I - P_{22}K$ is invertible, define $f(P, K) \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ by

$$f(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

The map $f(P, K)$ is called the (lower) ***linear fractional transformation*** (LFT) of P and K; we will also refer to this as the ***closed-loop map***. In the remainder of this note, we abbreviate our notation and define $G = P_{22}$.

B. Bounded Linear Operators

In this note, we consider the case where $\mathcal{U}, \mathcal{W}, \mathcal{Y}, \mathcal{Z}$ are Banach spaces and thus P is a bounded linear operator. We then introduce a little more notation.

For $S \subseteq \mathcal{X}$ and $T \subseteq \mathcal{X}^*$ define

$$S^\perp = \{x^* \in \mathcal{X}^* \mid \langle x, x^* \rangle = 0, \text{ for all } x \in S\}$$
$$^\perp T = \{x \in \mathcal{X} \mid \langle x, x^* \rangle = 0, \text{ for all } x^* \in T\}$$

where $\mathcal{X}^*$ is the dual-space to $\mathcal{X}$.

Given $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, we define the set $M \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$ of controllers K such that $f(P, K)$ is well-defined by

$$M = \{K \in \mathcal{L}(\mathcal{Y}, \mathcal{U}) \mid (I - GK) \text{ is invertible}\}.$$

For any Banach space $\mathcal{X}$ and bounded linear operator $A \in \mathcal{L}(\mathcal{X})$ define the ***resolvent set*** $\rho(A)$ by $\rho(A) = \{\lambda \in \mathbb{C} \mid (\lambda I - A) \text{ is invertible}\}$ and the ***resolvent*** $R_A : \rho(A) \to \mathcal{L}(\mathcal{X})$ by $R_A(\lambda) = (\lambda I - A)^{-1}$ for all $\lambda \in \rho(A)$. We also define $\rho_{\mathrm{uc}}(A)$ to be the unbounded connected component of $\rho(A)$.

Note that $1 \in \rho(GK)$ for all $K \in M$, and define the subset $N \subseteq M$ by

$$N = \{K \in \mathcal{L}(\mathcal{Y},\mathcal{U}) \mid 1 \in \rho_{\mathrm{uc}}(GK)\}.$$

C. Problem Formulation

We can now formally state the problem we are to address in this note.

Given Banach spaces $\mathcal{U}, \mathcal{W}, \mathcal{Y}, \mathcal{Z}$, generalized plant $P \in \mathcal{L}(\mathcal{W} \times \mathcal{U}, \mathcal{Z} \times \mathcal{Y})$, and a subspace of admissible controllers $S \subseteq \mathcal{L}(\mathcal{Y},\mathcal{U})$, we would like to solve the following problem

$$\begin{array}{ll} \text{minimize} & \|f(P,K)\| \\ \text{subject to} & K \in M \\ & K \in S \end{array} \quad . \tag{1}$$

Here $\|\cdot\|$ is any norm on $\mathcal{L}(\mathcal{W},\mathcal{Z})$, chosen to encapsulate the control performance objectives, and S is chosen to represent the desired decentralization of the controller. We call the subspace S the ***information constraint***.

This problem is very general, in the sense that the signal spaces $\mathcal{U}, \mathcal{W}, \mathcal{Y}, \mathcal{Z}$ may be continuous-time, such as L_2, or discrete-time, such as ℓ_2, and the signals and systems may evolve over infinite time, with $\mathcal{U}, \mathcal{W}, \mathcal{Y}, \mathcal{Z}$ function spaces over $[0,\infty)$ or $\mathbb{Z}_+$, over finite time intervals, or may be spatio-temporal. Also, the norm on $\mathcal{L}(\mathcal{W},\mathcal{Z})$ may represent either a deterministic measure of performance, such as the induced norm, or a stochastic measure of performance, such as the $\mathcal{H}_2$ norm.

In [10] and [11], we instead considered the case where $\mathcal{U}, \mathcal{W}, \mathcal{Y}, \mathcal{Z}$ were extended spaces L_{2e} or ℓ_e. The plant and controller were similarly assumed to be linear and continuous, and were further assumed to be causal and time-invariant. However, that did not restrict the plant and controller to be bounded, as they are in this note. An important special case was that where $P \in \mathcal{R}_{\mathrm{sp}}$ and $S \subseteq \mathcal{R}_p$. For that case, we were interested in the additional constraint that K be internally stabilizing. In this note, our additional concern involves the existence of an inverse, or $K \in M$.

This problem is made substantially more difficult in general by the constraint that K lie in the subspace S. Without this constraint, the problem may be solved by a simple change of variables, as discussed later. Note that the cost function $\|f(P,K)\|$ is in general a non-convex function of K. Even for finite-dimensional spaces $\mathcal{U}, \mathcal{W}, \mathcal{Y}, \mathcal{Z}$, no computationally tractable approach is known for solving this problem for arbitrary P and S.

D. Change of Variables

We define the map $h : \mathcal{L}(\mathcal{U},\mathcal{Y}) \times \mathcal{L}(\mathcal{Y},\mathcal{U}) \to \mathcal{L}(\mathcal{Y},\mathcal{U})$ by

$$h(G,K) = -K(I - GK)^{-1}$$

for all G, K such that $I - GK$ is invertible. We will also make use of the notation $h_G(K) = h(G,K)$, which is then defined for all $K \in M$. Given $G \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, the map h_G is an involution on M, as stated in the following lemma.

Lemma 1: For any $G \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, the map h_G satisfies $\mathrm{image}(h_G) = M$, and $h_G : M \to M$ is a bijection, with $h_G \circ h_G = I$.

Proof: A straightforward calculation shows that for any $K \in M$, $h_G(h_G(K)) = K$. It is then immediate that $\mathrm{image}(h_G) = M$ and $h_G \circ h_G = I$. ■

This lemma is very useful, since we have

$$f(P,K) = P_{11} - P_{12} h_G(K) P_{21}.$$

Hence, we have the standard parametrization of all closed-loop maps which are achievable by bounded controllers K. This parametrization is related to the well-known *internal model principle* and Youla parametrization of stabilizing controllers. Now, we can reformulate (1) as the following equivalent optimization problem:

$$\begin{array}{ll} \text{minimize} & \|P_{11} - P_{12} Q P_{21}\| \\ \text{subject to} & Q \in M \\ & h_G(Q) \in S \end{array} \quad . \tag{2}$$

The closed-loop map is now affine in Q, and its norm is therefore a convex function of Q. If not for the last constraint, that is, the information constraint S, we could solve this problem to find Q, and then construct the optimal K for problem (1) via the transformation $K = h_G(Q)$.

Note that while we are not considering all $Q \in \mathcal{L}(\mathcal{Y},\mathcal{U})$, only those $Q \in M$, in many cases of practical interest M is dense in $\mathcal{L}(\mathcal{Y},\mathcal{U})$. We will further discuss eliminating this constraint in Section III-A.

However, we see that the information constraint prevents this problem from being easily solved. Specifically, the set

$$\{Q \in M;\ h_G(Q) \in S\}$$

is not convex in general. The main thrust of this note is to seek conditions under which this last constraint may be converted to a convex constraint.

II. Quadratic Invariance

We now turn to the main focus of this note, which is characterizing which constraint sets S are invariant under h_G and thus lead to tractable solutions for problem (1). In [9], a property called *quadratic invariance* was introduced.

Definition 2: Suppose $G \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, and $S \subseteq \mathcal{L}(\mathcal{Y},\mathcal{U})$. The set S is called ***quadratically invariant*** under G if

$$KGK \in S, \quad \text{for all } K \in S.$$

Note that, given G, we can define a quadratic map $\Psi : \mathcal{L}(\mathcal{Y},\mathcal{U}) \to \mathcal{L}(\mathcal{Y},\mathcal{U})$ by $\Psi(K) = KGK$. Then a set S is quadratically invariant if and only if S is an invariant set of Ψ; that is, $\Psi(S) \subseteq S$.

We give a general lemma about quadratically invariant subspaces.

Lemma 3: Suppose $G \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, and $S \subseteq \mathcal{L}(\mathcal{Y},\mathcal{U})$ is a subspace. If S is quadratically invariant under G, then

$$K(GK)^n \in S, \quad \text{for all } K \in S,\ n \in \mathbb{Z}_+.$$

Proof: We prove this by induction. By assumption, given $K \in S$, we have that $KGK \in S$. For the induction step, assume that $K(GK)^n \in S$ for some $n \in \mathbb{Z}_+$. Then

$$\begin{aligned} 2K(GK)^{n+1} = (K + K(GK)^n) G (K + K(GK)^n) \\ - KGK - K(GK)^{2n+1} \end{aligned}$$

and since all terms on the right-hand side of this equation are in S, we have $K(GK)^{n+1} \in S$. ■

Quadratic invariance is easy to verify for a given problem, and can also be used to explicitly delineate which problems are amenable to convex synthesis within a particular class. For a discussion of conditions for quadratic invariance among specific constraint classes, such as sparsity constraints, symmetric constraints, and delay constraints, see [9] and [10].

III. Invariance Under Feedback

Before proving our main result, we state the following preliminary lemmas regarding analyticity.

Lemma 4: Suppose $D \subseteq \mathbb{C}$ is an open set, $\mathcal{X}$ is a Banach space, and $f : D \to \mathcal{X}$ is analytic. Suppose that $x \in D$, and $f(y) = 0$ for all y in an open neighborhood of x. Then, $f(y) = 0$ for all y in the connected component of D containing x.

Proof: See, for example, [3, Th. 3.7]. ■

Lemma 5: Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $D \subseteq \mathbb{C}$ is an open set, and $A : \mathcal{X} \to \mathcal{Y}$ is a bounded linear operator. Suppose $q : D \to \mathcal{X}$ is analytic, and $r : D \to \mathcal{Y}$ is given by $r = A \circ q$. Then, r is analytic.

Proof: This is a straightforward consequence of the definitions. ■

Lemma 6: Suppose $K \in \mathcal{L}(\mathcal{Y},\mathcal{U})$, $G \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, and $\Gamma \in \mathcal{L}(\mathcal{Y},\mathcal{U})^*$. Define the function $q_\Gamma : \rho(GK) \to \mathbb{C}$ by

$$q_\Gamma(\lambda) = \langle K R_{GK}(\lambda), \Gamma\rangle.$$

Then, q_Γ is analytic.

Proof: Define the linear map $\gamma : \mathcal{L}(\mathcal{Y}) \to \mathbb{C}$ by

$$\gamma(G) = \langle KG, \Gamma\rangle, \quad \text{for all } G \in \mathcal{L}(\mathcal{Y}).$$

Clearly γ is bounded, since

$$\|\gamma(G)\| \le \|K\|\ \|\Gamma\|\ \|G\|, \quad \text{for all } G \in \mathcal{L}(\mathcal{Y}).$$

Further $q_\Gamma = \gamma \circ R_{GK}$, and the resolvent is analytic, hence by Lemma V, we have that q_Γ is analytic. ■

Main Result: The following is the main result of this note. It states that given G, if a certain technical condition holds, then the constraint set S is quadratically invariant if and only if the information constraints on K are equivalent to the same *affine constraints* on the parameter $Q = h_G(K)$. In other words, subject to technical conditions, quadratic invariance is equivalent to invariance under feedback.

Theorem 7: Suppose $G \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, and $S \subseteq \mathcal{L}(\mathcal{Y},\mathcal{U})$ is a closed subspace. Further suppose $N \cap S = M \cap S$. Then

$$S \ \text{ is quadratically invariant under } G \iff h_G(S \cap M) = S \cap M.$$

Proof: ($\Longrightarrow$) Suppose $K \in S \cap M$. We first show that $h_G(K) \in S \cap M$. For any $\Gamma \in S^\perp$, define the function $q_\Gamma : \rho(GK) \to \mathbb{C}$ by

$$q_\Gamma(\lambda) = \langle K(\lambda I - GK)^{-1}, \Gamma\rangle.$$

For any λ such that $|\lambda| > \|GK\|$, the Neumann series expansion for R_{GK} gives

$$K(\lambda I - GK)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} K(GK)^n.$$

By Lemma 3, we have $K(GK)^n \in S$ for all $n \in \mathbb{Z}_+$ and, hence, $K(\lambda I - GK)^{-1} \in S$ since S is a closed subspace. Thus

$$q_\Gamma(\lambda) = 0, \quad \text{for all } \lambda \text{ such that } |\lambda| > \|GK\|.$$

By Lemma 6, the function q_Γ is analytic, and since $\lambda \in \rho_{\rm uc}(GK)$ for all $|\lambda| > \|GK\|$, by Lemma 4, we have

$$q_\Gamma(\lambda) = 0, \quad \text{for all } \lambda \in \rho_{\rm uc}(GK).$$

It follows from $K \in N$ that $1 \in \rho_{\rm uc}(GK)$ and, therefore, $q_\Gamma(1) = 0$. Hence

$$\langle K(I - GK)^{-1}, \Gamma\rangle = 0, \quad \text{for all } \Gamma \in S^\perp.$$

This implies

$$K(I - GK)^{-1} \ \in {}^\perp(S^\perp).$$

Since S is a closed subspace, we have ${}^\perp(S^\perp) = S$ (see, for example, [6, p. 118]) and, hence, we have shown $K \in S \cap M \implies h_G(K) \in S$. Since h is a bijective involution on M, it follows that $h_G(S \cap M) = S \cap M$ which was the desired result.

($\Longleftarrow$) We now turn to the converse of this result. Suppose S is not quadratically invariant. Then, there exists $K_0 \notin S$, such that $K_0 G K_0 \in S$. We will construct $K \in S \cap M$ such that $h_G(K) \notin S \cap M$. Without loss of generality we may assume $\|K_0\| = 1$. Choose $\Gamma \in S^\perp$ with $\|\Gamma\| = 1$ such that

$$\beta = \langle K_0 G K_0, \Gamma\rangle \in \mathbb{R} \quad \text{and} \quad \beta > 0$$

and choose $\alpha \in \mathbb{R}$ such that

$$0 < \alpha < \frac{\beta}{\|G\|(\beta + \|G\|)}.$$

Let $K = \alpha K_0$. Then $\|GK\| < 1$, $K \in S \cap M$, and

$$\langle K(I - GK)^{-1}, \Gamma\rangle = \sum_{i=0}^{\infty} \langle K(GK)^i, \Gamma\rangle$$

where we have used the fact that the map γ defined in Lemma 6 is bounded. Hence

$$\begin{aligned}
|\langle K(I - GK)^{-1}, \Gamma\rangle| &= \left|\sum_{i=0}^{\infty} \langle K(GK)^i, \Gamma\rangle\right| \\
&= \left|\alpha^2\beta + \sum_{i=2}^{\infty} \langle K(GK)^i, \Gamma\rangle\right| \\
&\ge \alpha^2\beta - \alpha \sum_{i=2}^{\infty} \|G\|^i \alpha^i \\
&= \alpha^2 \left(\frac{\beta - \alpha\|G\|(\beta + \|G\|)}{1 - \alpha\|G\|}\right) \\
&> 0.
\end{aligned}$$

Hence, $K(I - GK)^{-1} \notin S$ as required. ■

There are many cases of interest where the technical condition above is automatically satisfied, specifically any case where the plant is such that the resolvent set of GK is always connected. This includes the case where G is compact, such as for any problem where the Banach spaces are finite dimensional, as shown in the following corollary.

Corollary 8: Suppose $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is compact and $S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$ is a closed subspace. Then

$$S \text{ is quadratically invariant under } G \iff h_G(S \cap M) = S \cap M.$$

Proof: This follows since if G is compact then GK is compact for any $K \in S$ and, hence, the spectrum of GK is countable, and so $N = M$. ■

A. Equivalent Problems

When the conditions of Theorem 7 are met, we have the following equivalent problem. K is optimal for problem (1) if and only if $K = h_G(Q)$ and Q is optimal for

$$\begin{array}{ll} \text{minimize} & \|P_{11} - P_{12} Q P_{21}\| \\ \text{subject to} & Q \in M \\ & Q \in S \end{array} . \tag{3}$$

We then consider further reducing the problem to the following convex optimization problem:

$$\begin{array}{ll} \text{minimize} & \|P_{11} - P_{12} Q P_{21}\| \\ \text{subject to} & Q \in S \end{array} . \tag{4}$$

If the minimum to this problem did occur for some $Q \notin M$, it would mean that the original problem approaches its optimum as $(I - GK)^{-1}$ blows up. Therefore, the constraint that $Q \in M$ is unnecessary when the original problem is well-posed.

The solution procedure is then to solve problem (4) with convex programming, and recover the optimal controller for the original problem (1) as $K = h_G(Q)$.

B. Violation

The technical conditions of Theorem 7 are automatically satisfied in many cases of interest, such as in Corollary 8. However, the conditions cannot always be eradicated. We show an example where the resolvent set is not connected, the technical conditions fail, and the main result no longer holds. This is meant to elucidate that the conditions are not merely for ease of proof, but that the result can actually fail in their absence.

Consider the space of two-sided sequences

$$\ell_2 = \left\{ (\ldots, x_{-1}, x_0, x_1, \ldots) \,\middle|\, x_i \in \mathbb{R}, \sum_{i=-\infty}^{\infty} x_i^2 < \infty \right\}.$$

Define $\ell_2^+ = \{x \in \ell_2 \mid x_i = 0, \text{ for all } i < 0\}$ and define the delay operator $D : \ell_2 \to \ell_2$ as $D(x)_i = x_{i-1}$. Let $\mathcal{Y} = \mathcal{U} = \ell_2$, let the plant be the identity $G = I$, and let S be the subspace of causal controllers

$$S = \{K \in \mathcal{L}(\ell_2) \mid K(y) \in \ell_2^+, \text{ for all } y \in \ell_2^+\}$$

such that S is clearly quadratically invariant under G. Now, consider $K = 2D \in S$; we have

$$\begin{aligned} (I - GK)^{-1} &= -\frac{1}{2} D^{-1} \left(I - \frac{1}{2} D^{-1} \right)^{-1} \\ &= -\sum_{k=1}^{\infty} \frac{1}{2^k} D^{-k} \end{aligned}$$

and so $K \in M$. Also, note that

$$\rho(GK) = \{\lambda \in \mathbb{C} \mid |\lambda| \neq 2\}$$

and hence $\rho_{\mathrm{uc}}(GK) = \{\lambda \in \mathbb{C} \mid |\lambda| > 2\}$, which implies that $K \notin N$. Finally

$$K(I - GK)^{-1} = -\sum_{k=0}^{\infty} \frac{1}{2^k} D^{-k} \notin S.$$

So we have $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, $S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$ is a closed subspace, and S is quadratically invariant under G, but $N \cap S \neq M \cap S$. We have then found a $K \in S \cap M$ such that $h_G(K) \notin S$, and so $h_G(S \cap M) \neq S \cap M$.

This shows that the technical conditions of Theorem 7 cannot be completely eradicated for arbitrary Banach spaces.

IV. Conclusion

We considered the problem of minimizing an arbitrary norm of a linear-fractional transformation of bounded linear operators subject to a subspace constraint on the controller. In Theorem 7, we showed that a simple algebraic condition, ***quadratic invariance***, is necessary and sufficient for these affine constraints on the controller to be preserved under the feedback map, provided that a technical condition is met. We further showed that this allows for convex synthesis of optimal decentralized controllers. An example was then provided showing how the result can fail if the technical conditions are not met.

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